THE MATHEMATICAL STUDY OF SOME QUEUEING MODELS WITH APPLICATIONS

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Dedicated

to

my

parents

CERTIFICATE

This is to certify that the work embodied in this thesis entitled `Mathematical Study of Some Queueing Models With Applications', being submitted by Anjana Solanki, for the award of the degree of Doctor of Philosophy to the Bundelkhand University, Jhansi, has been carried out under my supervision and guidance, that the work embodied has not been submitted elsewhere for the award of any other degree and is up to the mark both in it's academic contents and the quality of presentation.

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(ANJANA SOLANKI)

PREFACE

The present work is the outcome of the researches carried out by me in the field of `Queueing Theory', at the department of Mathematics & Statistics, Bundelkhand University, Jhansi, while engaged in teaching as a lecturer, in the department of Applied Science & Humanities, Bundelkhand Institutie of Engineering & Technology, Jhansi.

The work presented in the thesis is based on my following research papers.

1. The transient state analysis of the bulk service queueing system $E_k/M^{a,b}/1/N$, with limited waiting space. (under communication)

2. The steady state analysis of the queueing system $E_k/M^{a,b}/1/N$, with finite waiting space (Communicated for publication)

3. Bulk service queueing system $E_k/M^{a,b}/1/N$, with general arrival rate and finite space. (under communication)

4. A preemptive priority queue with general arrivals and limited waiting space.(under communication)

5. A preemptive priority queue with a general bulk service rule. (Communicated for publication)

This thesis consists of six chapters, each divided in to serveral sections (progressively numbered 1.1, 1.2, ... etc.) The results in the text have been numbered serially, section and chapter wise, e.g. (3.2.1.) means first result of second section of chapter third. The references cited have been arranged alphabetically and yearwise at the end of the thesis. The references are indicated in the introduction of the thesis by giving the year within the parentheses.

This work starts with the study of the single server bulk service queueing system with limited waiting space. The time dependent and the steady state value have been obtained for the various queue disciplines.

Chapter I deals with introductions and the brief historical survey of the work, done in the field of Queueing Theory, with special references to the work embodied in the thesis.

Chapter II is devoted to generalize and unify the study of bulk service queueing system, studied by kambo and Chaudhry [105]. In this chapter we have determined the busy period distribution and its Laplace transform with the help of generating function technique.

Chapter III is devoted to the study of the queueing system, studied by Easton and Chaudhry [42]. In this chapter we have analysed the steady state behaviour of the system. Imbedded Markov chain results and the moments queue lengths are also discussed.

Chapter IV, is devoted to the situation similar to that discussed by Kambo and Chaudhry [105]. The single server bulk service queueing system is considered in which the units entering the system join a single queue of limited size. The interarrival times of the units have a general distribution with the given density function. The Laplace transform of the general probability has obtained.

Chapter V is devoted to the study of a preemptive priority queue with general arrivals and limited waiting space, studied by K.N. Gaur [51]. The supplementary variable technique and the Laplace transform have been used in this chapter, various results are experessed in terms of the well known hypergeometric function ${}_1F_1$.

We have obtained the probability density function and its Laplace transform for the system and also for the two particular cases.

Chapter VI is devoted to the study of a preemptive priority queue studied by R. Sirasamy [178]. The transient

state behaviour of the model is discussed and mean queue length of priority and non-priority customers are obtained. In this chapter we have assumed that the capacity of the priority units may be at the most M while that of non-priority units may be infinite.

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(ANJANA SOLANKI)

CONTENTS

CHAPTER 1

INTRODUCTION

Queueing theory is one of the most important branches of modern probability theory applied in technology and management. As far as the future development of technology and management may be extrapolated from the past and present state of affairs the need for the deeper insight in to queueing theory will increase rapidly. It is hardly necessary to point out the many actual queueing situations encountered in every day life. Production lines, the theory of scheduling and transportation (both surface and air traffic), the design of automatic equipment such as telephone and telegraph exchanges, and perticularly the rapidly growing field of information handling and data processing are but a few fields in which queueing situations are encountered. To characterize it arises in every situation, in which a facility for common use is provided, where waiting and queueing may and usually do arise. The organisation and the performance of the facility on the one hand and the behaviour of the users determine the queueing system.

The theory of queues and its applications have expanded greatly since the days of the pioneers, the theory has attracted the interest of some very applied

mathematicians, some of whom (Neuts, Pollaczek, and Takacs, for example) are known mainly for their work of queues and closely related to while others (Such as Kendall, Khintchine, and Hindley) have done impartant work on queues, but are move widely known for work in other bulads.

The queueing theory had its origin in the investigations in to the problems of telephone traffic the pioneer work in this field was done by Agnes krarup erlang (1878-1928), an engineer at the copehagin Telephone Exchange, whose fundamental paper has apprared in (1909-1920). Some workers before Erlang had used the theory of probability in the telephone traffic studies, for example. Grinsled, and Iohannsen (1907). continuing this task on the application of the probability theory to the telephone congestion problems. Erlang solved problems which are now fundamental in queueing theory. An excellent account of his work has been given by Brochmeyer, et al. (1948). Up to 1950, most of the work on the queueing theory was done in connection with telephone problem alone. After 1950, with the emergence of `operational Research' as a well recognized discipline. It was observed that the work done in connection with the telephone problem had a much wider scope. The queueing theory has been studied extensively and has found applications in a wide variety of field such as machine maintenance, road traffic, aviation problems,

defence operations, inventory management, etc. It has been used to study physical phenomena like the semiconductor noise problem (Belf, 1958). Panico (1969), have given various applications of queueing theory to the business, economics and science. The theory has applications to the following scientific areas: Catalysis in a chemical reaction, filtration process, gas molecules going through a whole, hospitals and the demand for medical case, Nervous reaction, and such psychological stress and strain as neurosis.

In queueing theory we study problem of the following general description. A number of units (or customers) arrive at a service facility according to some probabilistic law and wait (or do not wait) for service in some known fashion. They are served by one or more servics according to a given service discipline, the service times being random variables governed by some probabilistic law. and depant after serivce.

In mathematical analysis, a queueing system is normally characterized by means of the following terminology due to kendall (1951).

1.1 <u>Source</u> :

A source is defined as a device or a group or devices from which units emanate and call for service. The source is said to be infinite or

finite depending upon whether the number of units in the source is infinite or finite.

1.2 The Input Process:

Let the units arrive at a service facility at times $t_0 < t_1 < t_2 < \ldots < t_k < \ldots$; then the interarrival times are $U_r = t_r - t_{r-1}$ $(r \geq 1)$ The random variables U_r are in general assumed to be statistically independent and their distribution, A(u) in called the interarrival time distribution or the input distribution. It may be mentioned here that the restriction of independence has been removed by Winsten (1959) and Mercer (1960) in their studies of late arrivals.

1.3 The Queue Discipline :

This is the rule that determines the formation of the queue and the take up of customers for service from those waiting.

The most common queue discipline is `First come, first served' (FCFS), according to which the units are served in order of their arrival. There are various other discipline, such as the random service, batch service, last come first served (LCFS) and priority-service discipline.

1.4 <u>Number of Channels</u>:

The service facility can have one or more channels. Queueing process with single channel are called single-channel or single server queueing process, while those with more than one channel are called multichannel or multiserver queueing process.

1.5 <u>The service mechanism</u>:

The time which elapses while a unit is being served is called its service time. Let the customers depart at the instants $\sigma_1, \sigma_2, \ldots, \sigma_k, \ldots;$ $k=1,2,3, \ldots$ If, however, there is a depart at time t=0, we may write $\sigma_0 = 0$ and call an initial departure instant. If V_k, $k=1,2,3, \ldots$, is the duration of the kth service interval, here V's are independent of U's and also mutually independent, with the common distribution function.

$$B(v) = P(V_k < v), \ 0 <= v < \quad , k=1,2,\ldots.$$

(It we can write d A(u) = a (u) du, db(v) = b(v), then a(u) and b(v) are called the input and the service time density functions respectively).

As shown above, a queueing system is described in probabilistic terms. So due to chance variations in the inter-arrival or service times or both the queue parameters such as the queue length waiting

time, etc., exhibit statistical fluctions, and such as the determination of the distribution of these parameters is of interest. In general the following three parameters are investigated.

1.5.1. <u>The Queue Length Distribution</u> :

It is the distribution of the Number of units waiting in the queue or present in the system. Let N (t) be the number of customers in the queueing system including those in service, if any, at any epoch `t'. We will sometimes refer to N(t) informally as the system length and sometimes called the system size distribution.

1.5.2. <u>The Waiting Time Distribution</u> :

It is the distribution of the time a customer has to wait in the queue for getting service. Sometimes the service time is also included in the waiting time, and is called the customers's time in the system or `Sojourn time' Some authors, for example, Cox and Smith (1961) use the terms queueing time and waiting times according to the service time as excluded or included. The waiting time is important from the customers, point of view.

1.5.3. <u>The Busy Period Distribution</u> :

A busy period starts with the arrival of the customer who finds the system empty and ends when

the system next become empty. The distribution of such periods is called the busy period distribution and is important from the server's point of view.

Most of the earlier work on queueing theory was confined to the situation when the process has reached a state of equilibrium after the lapse of an indefinitily long period of time. The solutions thus obtained are called steady state solutions, However, the study of the temporal development of the process is important from practical as well as theoritical point of view and during recent years much work has been done on the time dependent or transient solution, from which steady state solutions can be deduced but letting the time t->
We shall use the following notations due to Kendall (1953) for the characterization of the various queueing systems we shall use

(i) M (Standing for Markov) for Poisson input or negative exponential service time distribution:

$A(u) = 1 - e^{-\lambda u} \quad (o<=u<\infty)$

Or $B(v) = 1-e^{-\lambda v} \quad (o<=v<\infty)$

$1/\lambda$ and $1/\mu$ are the mean inter-arrival and service times respectively.

(ii) D for deterministic or regular arrivals or service

$A(u) = 0 \ (u<1/\lambda) = 1 \ (u>=1/\lambda)$

Similarly for $B(v)$.

(iii) E_k for K- Erlang distribibution :

$$dA(u) = \frac{\lambda^K e^{-\lambda u} u^{K-1}}{(K-1)!} du, \qquad (0 <= u < \infty).$$

Similarly for $B(v)$. In the perticuler case when k=1 it reduces to exponential distribution and when $k \to \infty$, we get the rectangular distribution.

(iv) G for general or arbitrary sercive time distribuition.

(v) GI for general independent input or inter-arrival time distribution.

(vi) E_k for Erlang type k (k=1,2,...)

Any queueing system is described by giving first the input process, then the service time distribution, and finally the number of server's. For example, $M/E_k/1$ stands for a single server queueing system with Poisson arrivals and k- Erlang service time distribution, and GI/G/S stands for the most general case of general independent input, general service times and S servers.

Prior to 1950, most of the work was done or the assumption that the arrival or service time distributions are either exponential or k-Erlang or regular. Very little work was done for general distributions. Notable workers before 1950 are Erlang (M/M/S, M/D/S; $M/E_k/1$), Vacilot (M/M/S), Molina (M/M/S, M/D/S), Palm (M/M/S), Crommelin

(M/D/S, M/G/S/), Knintchine (M/D/1, M/G/1), Pollaczek (M/G/1, E_k/G/1, M/M/S, M/D/S, M/H/S), Borel (M/D/1) and Volverg (E_k/G/1). A review of the work done during this period has been given by Kendall (1951, 1953) and Saaty (1957).

Let us now define a different type of queueing processes.

1.6 <u>Stochastic Process</u> :

A Stochastic process or a random process is a process whose state is changing with time (or with regard to any other perameter) in a manner which involves probability.

A stochastic process is also said to be Markovian process.

1.7 <u>Markov Process</u> :

(After the Russian mathematician A.A. Markov)

If a knowledge of its present state is sufficient for prediction of its future, i.e. if the future states depend only on the present state and are independent of how the present state was attained. (A good accout of the theory and application of Markov process has been given by Bharucha- Reed (1960) non . Process which do not posses the Markovian Property are said to be non- Markovian. If in a Markovian processes we can find some

discrete points at which the states of the process posses the Markov property. we get the Markov chain imbedded in the process and such time points are called regeneration points of the process.

As an example, we may consider the queue length process of the M/M/1 queue. If the queue length at time t is n, then at time $t + \Delta$, the system will be in state n + 1 with probability $\mu\Delta+o(\Delta)$, in state n-1 with Probability $\mu\Delta+o(\Delta)$ and in state n with probability $1-(\lambda+\mu) + o(\Delta)$, Where and are the mean arrival and service rates. Thus a knowledge of the state of the system at time t is sufficient to predict its Probabilistic behaviour at the time $t+\Delta$, and the process is Markovian, The same is true for the system M/M/S.

Most of the earlier work on the subject has been done for Poisson queue's, that in those in which the arrival distribution is Poisson and service distribution in negative exponential. Now we shall describe some important techniques of solving queueing problems with general distributions.

1.8 Imbedded Markov Chain Technique :

The queue length process for the queueing system M/M/1 is Markovian. This is so because the exponential distribution posscsses the `forget

fulness' property. That is, in the case of exponential service times, the probability of a service completion in time $(t, t + \Delta)$ does not depend on the elapsed service time of the unit being served this is not true for the system M/G/1.

Markov chains are a subclass of Markov process in which the state space is discrete (Finite or Countabley infinite), but the time parameter may be discrete or continuous. Essentially a Markov process is a Markov chain iff the state space is discrete. The bulk arrival poisson process is a continuous Markov chain. Kendall (1951) observed that the system M/4/1 the epochs when the customers depart after service are points of regencration and thus the queue length at these epochs constitute as imbedded Markov chain.

Kendall (1953) used the same method for the Process GI/M/S, for which the arrival epochs are the regeneration points.

This was the first systematic method of solving queueing system with general distributions and has therefore been widely used by subsequents workers Wishart (1956) used it for the more general $GI/E_k/1$ and later (1959) extended it to the system GI/L/1 (Where L is used to denote the class of

distributions that can be simulated by the use of phases: see `Phase technique'). Takacs (1958) and Finch (1959) applied this technique in their studies of queues with general arrival and finite waiting space Bailey (1954), Miller (1959), Jaiswal (1961) and Bhat (1964) used it to study bulk queues, Srinivasan and Subramaniam (1969) and Srinivasan and Kalpakam (1971) have also used this method.

Gaver (1959, 1962) used an extention of Kendall,s method which have been called the `extended chain method' by Keilsor (1964). The method consists in corisidering a two dimensiticnal Markov chain, the two companents specifying the queue length at a regeneration point and the time of regeneration. This is suitable for continuous time treatment.

1.9 Supplementary Variable Technique:

This method consists in enhancing the description of the states of the system by including one or more supplementary variables so as to make the process Markovian. Thus the queueing process M/G/1 is made Markovian by including the variable x, the elapsed service time of the unit being serviced, and we define Pn(x, t) dx as the probability that at time t the queue length is n and the elapsed

service time of the unit under service lies between x and x + dx. Similarly the process GI/M/1 may be dealt with the inclurion of the elapsed time since the last arrival.

This technique was used in 1942 by kendall. The name appears due to cox (1955). Kendall (1953) considered this technique, who called it the `alligmentation technique', gave an abstract formulation of the method and mentioned that the most general process GI/G/S can be studied similarly by using S+1 Supplementary variables denoting respectively the elapsed time since the last arrival and the elapsed service time of each of the S servers.

Cox (1955) used this method to formulate equatirns for the system M/G/S and gave the steady state soluction for the case M/G/1 using this technique, Klilsor and kooharian (1960) obtained the time dependent solution of the process GI/G/1 by reducing it to reilbert problem. Wishart (1960) used the unexpended service time as the supplimentary variable to study the queueing process M/G/1 and GI/M/1 under queue discipline last come, first served. Conolly (1958, 1959, 1960) used this technique in his studies of the process

GI/M/1 and (1960) to study the system $GI/E_k/1$.

Jaiswal (1961, 1962) applied the technique to obtain transient solutions of the preemptive resume and the head-of-the-line priority queues. The same technique has been used by Thiruvengadam (1963, 1964) to study queueing systems subject to breakdown, by Thiruvengadam and Jaiswal (1963, 1964) to study machine interference problems and by Jaiswal (1965) to study the M/G/1 queue with talking and reneging. Kashyap (1965, 1965, 1967) used the supplementary variable technique to study the transient behaviour or the double ended queue with limited waiting space and in (1966) to study the same queueing system under the bulk service discipline. Hawkes (1966) extending the used of supplementary variable technique has studied head-of- the- line priority queue with bulk arrivals Gaur and Kastyap (1970, 1972) used the supplementary variable technique to study preemptive repeat priovity and preemptive queueing discipline, later (1972) to study the time-deperdent solution of preemptive- repeat priority queueing discipline with bulk arrivals.

Gaur and Kashyap (1972), using the supplementary variable method obtained the joint distribution of

the queue length and the number served in time t in an M/G/1 queue, later (1972), using the same method, the head- of- the- line priority queueing system with general service has been studied to obtain an expression for the L.T. of the generating function of the joint probability distribution of the number of priority and nonpriority units in the queue at time t, and the number of units served (including non- priority units) in time t.

1.10 Phase Technique:

This technique is essentially due to Erlang. Erlang obtained a generalization of the exponential distibution regarding the service channel as consisting of K fictitious phases, such that the service in each phase in exponential with mean rate u. The service on a unit is taken to commence with its entry into the first phase and to last until the unit has left the last phase; then the next unit in the queue enters the channel at the first phase. The service time distribution thus obtained in the K-fold convolution of the exponential distribution with itself and is known as the K-Erlang distribution.

Erlang distributions provide a family of service-time distributions which range all the way

from the `pure random' exponential type to the completely deterministic constant service time distributions. They will not fit all possible service-time distribution but they will fit many (and perhapes most) many of the ones encountered in Practice so when K=1 it reduces to exponential distribution, and to the rectangular distribution (Constant service times) when K= . Similarly, we define a j-Erlang arrival distribution by considering an arrival-timing channel consisting of `j' fictitious phases. Morse (1958) has used these distribution to solve many queueing problem.

Cox (1955) showed that the method can be applied to any distribution having a rational Laplace transform by the use of complex transition probabilities, Gaver (1954) modified Erlang's method by assuming the exsisterice of an infinite number of phases such that an arriving customer demand r phases of service with probability Cr, where O<=Cr<=1 and Cr=1, and showed that by choosing Cr suitably, different distributions can be obtained.

Luchak (1956) showed that both from theoretical and practical points of view it is better to take a finite number j of phases, so that now r=1, Cr=1,

and then in order to simulate any given distribution we have only to determine the Cr's and the paraments of the exponential phases. In particular, K-Erlag distribution corresponds to the case $Cr = {}_{r,k}$, the kronecter delta. The phase technique, though an approximate one, is particularly suitable for computational purposes.

One disadvantage with the technique was that the equations were formulated in terms of the number of phases in the system instead of the number of customers so that the phase length distributions could be obtained instead of the queue legth distribution. Jaiswal (1960) has removed this disadvantage by defining $P_{n,r}(t)$ as the probability that a time t there are n units in the queue and the service is in the r^{th} phase.

Jaiswal used this method to study the bulk service queue with variable capacity (1961) and the GI/M/1 queue with limited waiting space (1961). Kashyap (1965) used this technique combined with the supplementary variable technique to study queues. Gaur and kashyap (1972) using the phase technique studied the double ended queueing model with batch departures by using this technique.

Gupta (1965) studied a simple queue with limited waiting space taking a more generalized service distribution termed as mixed-Erlangian service time distribution. The mixed-Erlangian distribution differs from the Erlangian distribution in the sense that here the mean times of stay in the various phases which comprise the arrival of service channel are not the same, the time a unit stays in the r^{th} phase is distributed according to the negative exponential distributions with mean value $1/\ _r$ (instead of 1/ as in Erlang distribution). Gaur and Kashyap (1972) have studied a double- ended queue with mixed- Erlangian arrival distribution and limited waiting space.

1.11 Semi Markov Method:

This method is again a modification of the imbedded Markov chain technique. A semi Morkov process is a process X (t) such that the state Xn after the nth transition is a Markov chain. but the length of time T spint by X (t) in a given state has an arbitrary distribution. Fabens (1961) used this technique to study the system $E_s/G/1$ and the system M/G/1 with service in batches exactly of size S. Semi-Markov processes have been extensively studied by Levy (1954). Smith (1955) and Pyke (1961, 1963), some recent work on the application of semi-Markov

process of queueing theory is due to Neuts (1964, 1966), Nair (1971), Itall and Disney (1971) and Neuts and shun-Zerchen (1972).

1.12 Integral Equation Approach :

The work referred to above has been done by regarding the queue length, which is a discrete variable, as the stochastic variable of the process. Some authors have, hawever approached queueing problem via the conisideration of waiting time a continuous variable. Lindley (1952) investigated the waiting time distribution for the general- single- server queueing system GI/G/1 and obtaind the limited distribution of waiting times as the solution of an integral equation of the wiener- Hopf type. Lindley gave the solution of the integral equation for the process M/G/1 and $D/E_k/1$,

Smith (1953) considered the general solution of the equation and showed that the waiting time distribution for the system GI/M/1 is exponential under Certain restrictions on the arrival distribution. Syshi (1960) gave the solution of Lindley's equation for the system $E_n/E_m/1$.

Takacs (1955) gave his well known integro differential equaltion for the distribution function of the virtual waiting time (t) that is

the waiting time of a customer arriving at time t, for the queueing system M/G/1 with time dependent arrival rate (t). Takac's analysis was extended by Benes (1957), 1960), Reich (1958, 1959), and Takacs himself (1962, 1963). The virtual waiting time process has been extensively studied by Takacs (1962).

The integral equation approach has been used extensively by Prabha (1960, 1962, 1964, 1967, 1969). Prabha and Bhat (1963, 1963) and by Bhat (1964, 1965, 1967, 1969, 1972) for different queueing models. More literature on the approach can be found in Prabhu (1965) and Bhat (1968).

We shall concluds by giving some remarks about the `double-ended queue' and the priority queueing disciplines.

1.13 Double-Ended Queueing System :

It is a queueing system in which the waiting customers and idle servers both form queues. As an example, we may consider the queues of customers and taxis at a taxi- stand where at any time there is either a queue of customer or one of taxis or the taxi-Stand is totally empty. An inventory system, where at times there is a queue of the items stored, while at other times there is a queue

of backlog orders (when back orders are also allowed), it is an another example of it.

Kendall (1951) studied the double endend queueing system with poisson arrivals of customers and taxi with infinite waiting space for both customer are taxi. Dobbie (1961) studiled the same system, giving a generalization to the case of time-dependent Parameters. Sasieni (1961) gave the steady state solution for the double-ended queue with state dependent parameters in which there in limited waiting space for M taxis or N customers and the taxi and customers do not wait for langer than times S and T respectively, application of this model of inventory problems was also discussed. Jain (1962) gave the time dependent solution for the double ended queue with Poisson arrivals of customers and general arrivals of taxis and with a finite waiting space for taxis only.

Kashyap (1965) studied the randam walk model which leads to the transient solution of double-ended queue with Poisson arrivals of customers and taxi with finite waiting space for both taxis and customers.

Kashyap (1965, 1967) studied the generalization of the system to the case of general arrival of taxis

and later (1966, 1969) extended it to the bulk service case with S-seated taxis.

Gaur and Kashyap (1972) studied the double-ended queueing system with limited waiting space for both customers and taxis, with poisson arrivals of customers, while arrival distribution of taxis is simulated by mean of phase technique, and later (1972) the system was studied for mixed-Erlangian arrival distribution for taxis Gaur and kashyap (1972) Considered the double-ended queueing model where an arriving taxi departs of taking a variable number of customers, while arrival distributions of customers is simulated by mean of phase method, and later (1972) studied the double ended queueing system where the arrival rates for taxis as well as for customer have been taken to be functions of number of units already present in the queue,; applications to inventory control have been suggested.

1.14 Priority Queueing Discipline :

The most connmon queue discipline is the so called `first come, first served' discipline, according to which the units are served in order of their arrival. There are a number of other queue disciplines of interest for many actual queueing

situation. These disciplines concern of choice of the next customer to be served when the server terminate, a service. For example, units may be takes up for service at random, or `the last Come first served' or on priority basis.

Suppose arriving customers are classified into two different types. If waiting or arriving customers of type always have priority for service over those of type 2, while customers of the same type are served in order of arrival, it is said that the single server queue operates according to a priority discipline with two levels.

Any priority discipline must, therefore, specify the rules making the following two decision :

(a) When the server tcminates a service, which type of unit is to be taken for service.

(b) Whether to continue or discontinue the service of the unit being serviced on the arrival of a priority unit, the decision to select the next unit for service may lead to the following priority discipline :

1.14.1 <u>Preemptive</u> :

The service of the type-2 unit is immediately interrupted and the server starts serving the type-1 unit.

1.14.2 <u>Head-Of-The-Line</u>:

The service of a type-2 units is continued to completion. This is also called postponable or non-preemptive priority.

1.14.3 <u>Discrationary</u>:

The servers may use the discretion to follow (i) or (ii) depending upon the elapsed service time of the type 2 unit.

The preemptive discipline can be further divided in to the following categories:

(1) **Preemptive resume:** The preemptive unit resumes service from the point where it was interrupted.

(2) **Preemptive Repeat-Identical:** The preemptive unit on its re-entry requires the same amount of service as it required on its earlier entry.

(3) **Preemptive Repeat-Different:** The preemptive unit on its re-entry requires a random service time independent of past preemptions and wasted service time.

The disciplines discssed in (i) and (ii) are termed as exogeneous priority disciplines, whereas

disciplines of type (iii) In which decision of selecting the next unit for service may be based on other considerations e.g. the type of unit last served or the waiting time of the unit present are called endogeneous priority disciplines. An example where such a discipline exists is the situation at a controlled traffic intersection when one stream of vehicles is allowed to pass as long at there are vehicles in that stream, and then vehicles from another stream are allowed to pass till they are available, in this way the cycle goes on. So the priority alternates from class 1 to class 2 and is therefore called an alternating priority discipline, the generatization to the case K>2 is quite obvious and also practicable.

Cabham (1954) studied a K-class queueing process with head- of- the the line priority discipline and obtained expressions for average steady state waiting times. White and Christie (1958) studies the preemptive resume priority queueing with exponential arrival and service times for the steady state case. Heathcote (1959, 1960, 1961) investigated preemptive priority queueing problems for time dependent case Miller (1960) studied the head-of-the-line priority queueing with general service time distribution by using the imfedded

Markov chain technique. Gaver (1962) and Keilson (1962) have studied the preemptive priority queueing (including preemptive-repeat) with interrupted service for general service time distribution, using the nation of `Completion time'. Takacs (1964) obtained expressions for the stationary distribution of the waiting time for the priority levels, (i) service with privileged interruption, (ii) service without interruption. Avi- Itzhak (1963) studied preemptive- repeat priority queue as a special case of the multipurpose server problem.

Jaiswal (1961), using the supplementary variable technique, studied the preemptive resume priority queue problem for general service time distribution. Jaiswal (1962) obtained the Laplace transform of the time-dependent probabilities in a head-of-the-line priority queue characterized by Poisson arrivals and general service time distributions, using the method of supplementary variable. Jaiswal and Thiruvengadam (1963) using the supplementary variable technique have discussed simple machine interference two types of failure as a priority queueing model, with Hawkes (1966) extended Jaiswal's (1962) queueing problem for bulk arrivals using the same method.

Avi-Itzhak, Maxwell and Miller (1965) have studies a 2-class queueing system with alternating priorities with the help of busy period process and then obtained general processes probabilities by using renewal behaviour of the queue with alternating priorities with investigated 2-class queueing process with alternating plane, taking different switch rules, Hooks (1972) studied a priority queue in which arrivals for the low priority class is taken to be general.

Gaur and Kashyap (1972), using the supplementary variable technique, studied the busy period in a preemptive repeat queueing with general service time distribution; later in (1972) we have considered a preemptive queueing discipline with general arrivals and limited waiting space for both priority and non- priority units. Gaur and Kashyap (1972) studied the preemptive repeat discipline with general service time-distributions and assuming arrivals of both types of units to be in batches of variable size. Using the supplementary variable technique Laplace transforms of the generating functions of the joint probability distribution of queue-lengths have been obtained.

The work of Kambo and Chaudhry [105], Eston and Chaudhry [42], K.N. Gaur [51], B.R.K. Kashyap [107], R. Sivasamy [178], have also been used in this thesis and some of them have been the cause of motivation of some of the chapters.

Much work has been done in recent years on many varied aspect of the subject of queueing theory. A vast amount of literature, when is still growing rapidly, exists on this subject. In the review given above, it has not been possible to cover more than a small proportion of the huge amount of material available.

CHAPTER 2

2.1 INTRODUCTION

The queueing system $E_k/M^{a,b}/1/N$ with finite waiting space is the modification of the system studied by Kambo and Chadhry [105] with infinite waiting space. Bulk service is governed by a general rule : service begins only when 'a' units (quorum) are present. If the queue length is $\geq$ a but $\leq$ b then entire queue is taken up for service; if there are more than 'b' units in queue then the server accepts first 'b' units. This rule was first discussed by Neuts [146] and then by Holman [90] and others. Borthakur [23] studied the busy-period distribution for the system $M^x/G^{a,b}/1$ and obtain its Laplace transform.

Our interest here is to discuss the distribution of the busy period for the same system with finite waiting space. We obtain the Laplace transform of p.d.f. of the busy period.

This chapter deals with the distribution of the busy period for a bulk service queueing system wherein interarrival times follow an Erlangian distribution. Using the Erlangian technique the busy-period equations for the single server bulk service system $E_k/M^{a,b}/1/N$ are solved.

However, since $f_r^*(0) \neq 1$, $f_r^*(s)$ can not be the transform of the p.d.f. of the busy period. In fact, the transform of the busy period p.d.f. should be $\sum_{r=0}^{a-1} f_r^*(s)$.

If this correction in Borthakur's result is made, inversion and expressions for the moments for the special cases of this model become much simpler.

In 2.2 of this chapter we discuss the busy period equations for the system and obtain the different probabilities. In 2.3 we discuss the busy period of the system. In 2.4 the computer programme for the sample listings of the root α of the characteristic equation has been discussed and these listings are also presented in the tabular form.

2.2 BUSY PERIOD EQUATIONS

Define

$\bar{P}_{n,r,1}(t)$ = Probability that at time $t(\geq 0)$, the queue size is $n(\geq 0)$, the arrival is in phase $r(1 \leq r \leq k)$, and the server busy;

$P_{n,r,0}(t)$ = Probability that at time t the queue size is $n(0 \leq n \leq a-1)$ the arrival is in phase $r(1 \leq r \leq k)$, and the server is idle.

Relating the probabilities at time $t + \Delta\ t$ to those at time t , we get the following transition equations for the system :

$$P_{0,1,0}(t+\Delta t)=P_{0,1,0}(t)(1-\mu\ \Delta\ t)(1-k\ \lambda\ \Delta\ t)$$

$$+\mu\Delta t(1-k\ \lambda\Delta t)Q_{0,1,1}(t)$$

$$+\mu\Delta tP_{0,1,0}(t), \qquad \ldots(2.2.1.)$$

$$P_{n,1,0}(t+\Delta t) = P_{n,1,0}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+\mu\Delta t\ P_{n,1,1}(t)$$

$$+\mu\Delta t(1-k\ \lambda\Delta t)P_{n,1,1}(t)+k\ \lambda\Delta t\ P_{n-1,k,0}(t),$$

$$(1\leq n\leq a-1) \qquad \ldots(2.2.2.)$$

$$P_{n,r,0}(t+\Delta t) = P_{n,r,0}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+\mu\Delta tP_{n,r,1}(t)$$

$$+\mu\Delta t(1-k\ \lambda\Delta t)P_{n,r,1}(t)$$

$$+k\ \lambda\Delta t(1-\mu\Delta t)P_{n,r-1,0}(t),$$

$$(0\leq n\leq a-1\ ;\ 2\leq r\leq k) \qquad \ldots(2.2.3.)$$

$$P_{0,1,1}(t+\Delta t) = P_{0,1,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+k\ \lambda\Delta t\ P_{a-1,k,0}(t)$$

$$+\mu\Delta t\ \sum_{j=a}^{b} P_{j,1,1}(t), \qquad \ldots(2.2.4.)$$

$$P_{0,r,1}(t+\Delta t) =P_{0,r,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)$$

$$+k\ \lambda\Delta tP_{0,r-1,1}(t)\ +\mu\Delta t\ \sum_{j=a}^{b}\ P_{j,r,1}(t),$$

$$(2\leq r\leq k) \qquad \ldots(2.2.5.)$$

$$P_{n,1,1}(t+\Delta t) = P_{n,1,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+k\ \lambda\Delta t P_{n-1,k,1}(t)$$

$$+\mu\Delta t\ P_{n+b,1,1}(t),$$

$$(1 \le n \le N-b) \qquad \ldots(2.2.6.)$$

$$P_{n,r,1}(t+\Delta t) = P_{n,r,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+k\ \lambda\Delta t P_{n,r-1,1}(t)$$

$$+\mu\Delta t\ P_{n+b,r,1}(t),$$

$$(1 \le n \le N-b\ ;\ 2 \le r \le k) \qquad \ldots(2.2.7.)$$

$$P_{n,r,1}(t+\Delta t) = P_{n,r,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+k\ \lambda\Delta t\ P_{n,r-1,1}(t),$$

$$(N-b+1 \le n \le N\ ;\ 2 \le r \le k) \qquad \ldots(2.2.8.)$$

$$P_{n,1,1}(t+\Delta t) = P_{n,1,1}(t)(1-\mu\Delta t)(1-k\ \lambda\Delta t)+k\ \lambda\Delta t\ P_{n-1,k,1}(t).$$

$$(N-b+1 \le n \le N) \qquad \ldots(2.2.9.)$$

Transposing and letting $\Delta t \to 0$, we have

$$\frac{d}{dt} P_{0,1,0}(t) = -K\ \lambda P_{0,1,0}(t)+\mu Q_{0,1,1}(t), \qquad \ldots(2.2.10.)$$

$$\frac{d}{dt} P_{n,1,0}(t) = -K\ \lambda P_{n,1,0}(t)+\mu P_{n,1,1}(t)+K\ \lambda P_{n-1,k,0}(t),$$

$$(1 \le n \le a-1) \qquad \ldots(2.2.11.)$$

$$\frac{d}{dt} P_{n,r,0}(t) = -K\lambda P_{n,r,0}(t)+\mu P_{n,r,1}(t)+K\lambda P_{n,r-1,0}(t),$$

$$(0\le n\le a-1\ ;\ 2\le r\le k) \qquad \ldots(2.2.12.)$$

$$\frac{d}{dt} P_{0,1,1}(t) = -(K\lambda+\mu)P_{0,1,1}(t)+K\lambda P_{a-1,k,0}(t)$$

$$+\mu \sum_{j=a}^{b} P_{j,1,1}(t), \qquad \ldots(2.2.13.)$$

$$\frac{d}{dt} P_{0,r,1}(t) = -(K\lambda+\mu)P_{0,r,1}(t)+K\lambda P_{0,r-1,1}(t)$$

$$+\mu \sum_{j=a}^{b} P_{j,r,1}(t),$$

$$(2\le r\le k) \qquad \ldots(2.2.14.)$$

$$\frac{d}{dt}P_{n,1,1}(t)=-(K\lambda+\mu)P_{n,1,1}(t)+K\lambda P_{n-1,k,1}(t)+\mu P_{n+b,1,1}(t),$$

$$(1\le n\le N-b) \qquad \ldots(2.2.15.)$$

$$\frac{d}{dt} P_{n,r,1}(t) = -(k\lambda+\mu)P_{n,r,1}(t)+k\lambda P_{n,r-1,1}(t)$$

$$+\mu P_{n+b,r,1}(t),$$

$$(1\le n\le N-b;\ 2\le r\le k) \qquad \ldots(2.2.16.)$$

$$\frac{d}{dt} P_{n,r,1}(t) = -(k\lambda+\mu)P_{n,r,1}(t)+k\lambda P_{n,r-1,1}(t),$$

$$(N-b+1\leq n\leq N \ ; \ 2\leq r\leq k) \qquad \ldots(2.2.17.)$$

$$\frac{d}{dt} P_{n,1,1}(t) = -(k\lambda+\mu)P_{n,1,1}(t)+k\lambda P_{n-1,k,1}(t).$$

$$(N-b+1\leq n\leq N) \qquad \ldots(2.2.18.)$$

Define the following probability generating functions :

$$P_{n,1}(y,t) = \sum_{r=1}^{k} P_{n,r,1}(t)\, y^{r}, \qquad \ldots(2.2.19.)$$

$$P_{1}(z,y,t) = \sum_{n=0}^{N} P_{n,1}(y,t)z^{n} \ ; \ |z|<1. \qquad \ldots(2.2.20.)$$

Following the asual technique of generating function analysis, from (2.2.19.) & (2.2.20.), we have

$$\frac{d}{dt} P_{1}(z,y,t) = \sum_{n=0}^{N} \sum_{r=1}^{k} \frac{d}{dt} P_{n,r,1}(t)y^{r}z^{n}. \qquad \ldots(2.2.21.)$$

$$\frac{d}{dt} P_{1}(z,y,t) = \sum_{r=1}^{k} \frac{d}{dt} P_{0,r,1}(t)Y^{r}+\sum_{n=1}^{N-b} z^{n} \sum_{r=1}^{k} \frac{d}{dt}P_{n,r,1}(t)y^{r}$$

$$+\sum_{n=N-b+1}^{N} z^{n}\sum_{r=1}^{k} \frac{d}{dt} P_{n,r,1}(t)y^{r}. \qquad \ldots(2.2.22.)$$

Now taking appropriate summation of each term in (2.2.22.) we get.

$$\sum_{r=1}^{k} \frac{d}{dt} P_{0,r,1}(t)y^r = -(k\lambda+\mu) \sum_{r=1}^{k} P_{0,r,1}(t)y^r + k\lambda P_{a-1,k,1}(t)$$
$$+k\lambda \sum_{r=2}^{k} P_{0,r-1,1}(t)y^r$$
$$+\mu \sum_{j=a}^{b} \sum_{r=1}^{k} P_{j,1,1}(t), \qquad \ldots(2.2.23.)$$

$$\sum_{n=1}^{N-b} z^n \sum_{r=1}^{k} y^r \frac{d}{dt} P_{n,r,1}(t) = \sum_{n=1}^{N-b} z^n \Bigg[-(k\lambda+\mu)\sum_{r=1}^{k} P_{n,r,1}(t)$$
$$+\mu\sum_{r=1}^{k} P_{n+b,r,1}(t)\, y^r + k\lambda y P_{n-1k,1}(t)$$
$$+k\lambda \sum_{r=2}^{k} P_{n,r-1,1} y^r \Bigg], \qquad \ldots(2.2.24.)$$

$$\sum_{n=N-b+1}^{N} z^n \sum_{r=1}^{k} y^r \frac{d}{dt} P_{n,r,1}(t) = \sum_{n=N-b+1}^{N} z^n \Bigg[-(k\lambda+\mu)\sum_{r=1}^{k} P_{n,1,1}(t)$$
$$+k\lambda y P_{n-1,k,1}(t) + k\lambda \sum_{r=2}^{k} y^r P_{n,r-1,1}(t) \Bigg]. \qquad \ldots(2.2.25.)$$

Putting the values from (2.2.23.)-(2.2.25.) in (2.2.22.) we have.

$$\frac{d}{dt} P_1(z,y,t) = -(k\lambda+\mu \sum_{n=0}^{N} z^n \sum_{r=1}^{k} P_{n,r,1}(t)y^r + k\lambda y P_{a-1,k,0}(t)$$

$$+\mu\sum_{n=0}^{b} z^n\sum_{r=1}^{k} P_{n,k,1}(t)y^r + k\lambda\sum_{n=0}^{N} z^n\sum_{r=2}^{k} P_{n,r-1,1}(t)y^r$$

$$+\mu\sum_{n=1}^{N-b} z^n\sum_{r=1}^{k} P_{n+b,r,1}(t)y^r$$

$$+k\lambda y\sum_{n=1}^{N} P_{n-1,k,1}(t)z^n \quad . \qquad \ldots(2.2.26.)$$

Obtaining the values of certain terms in (2.2.26.) by usual method we get

$$k\lambda\sum_{n=0}^{N} z^n\sum_{r=2}^{k} y^r \; P_{n,r-1,1} = k\lambda y P_1(z,y,t) - k\lambda Y^{k+1} P_{n,k,1}(t),$$

$$\ldots(2.2.27.)$$

$$\mu\sum_{n=1}^{N-b} z^n\sum_{r=1}^{k} P_{n+b,r,1}(t)y^r = \mu z^{-b} P_1(z,y,t) - \mu z^{-b}\sum_{n=0}^{b} P_{n,1}(y,t)z^n,$$

$$\ldots(2.2.28.)$$

$$k\lambda y\sum_{n=1}^{N} P_{n-1,k,1}(t)z^n = k\lambda yz\sum_{n=0}^{N} P_{n,k,1}(t)z^n - k\lambda yz^{N+1} P_{N,k,1}(t).$$

$$\ldots(2.2.29.)$$

Putting the values from (2.2.27.)-(2.2.29.) in (2.2.26.)we get.

$$\frac{d}{dt} P_1(z,y,t)=(k\ \lambda y-k\ \lambda-\mu+\mu z^{-b})P_1(z,y,t)-\mu z^{-b}\sum_{n=0}^{b} P_{n,1}(y,t)z^n$$

$$+\mu\sum_{n=a}^{b} P_{n,1}(y,t)+k\ \lambda y(z-Y^k)\sum_{n=0}^{N} P_{n,k,1}(t)z^n$$

$$-k\ \lambda yz^{N+1}P_{N,k,1}(t)+k\ \lambda yP_{a-1,k,0}(t).$$

...(2.2.30.)

Taking Laplace transforms of (2.2.10)-(2.2.18) and (2.2.30) we get

$$(S+K\ \lambda)\bar{P}_{0,1,0}(S) - \mu\bar{Q}_{0,1,1}(S) = 0, \qquad ...(2.2.31.)$$

$$(S+K\ \lambda)\bar{P}_{n,1,0}(S) - \mu\bar{P}_{n,1,1}(S)-k\ \lambda\bar{P}_{n-1,k,0}(S)=0,$$

$$(1\le n\le a-1) \qquad ...(2.2.32.)$$

$$(S+K\ \lambda)\bar{P}_{n,r,0}(S) - \mu\bar{P}_{n,r,1}(S)-k\ \lambda\bar{P}_{n,r-1,0}(S)=0$$

$$(0\le n\le a-1\ ;\ 2\le r\le k), \qquad ...(2.2.33.)$$

$$(S+K\ \lambda+\mu)\bar{P}_{0,1,1}(S) - k\ \lambda\bar{P}_{a-1,k,0}(S)-\mu\sum_{j=a}^{b} \bar{P}_{j,1,1}(S)=0,$$

...(2.2.34.)

$$(S+K\ \lambda+\mu)\bar{P}_{0,r,1}(S) - k\ \lambda\bar{P}_{0,r-1,1}(S)-\mu\sum_{j=a}^{b} P_{j,r,1}(S)=0,$$

$$(2\le r\le k) \qquad ...(2.2.35.)$$

$$(S+K\lambda+\mu)\overline{P}_{n,1,1}(S) - k\lambda\overline{P}_{n-1,k,1}(S)-\mu\overline{P}_{n+b,1,1}(S),$$

$$(1\leq n\leq N-b) \qquad \ldots(2.2.36.)$$

$$(S+K\lambda+\mu)\overline{P}_{n,r,1}(S) - k\lambda\overline{P}_{n,r-1,1}(S)-\mu\overline{P}_{n+b,r,1}(S),$$

$$(1\leq n\leq N-b \;;\; 2\leq r\leq k) \qquad \ldots(2.2.37.)$$

$$(S+K\lambda+\mu)\overline{P}_{n,r,1}(S) - k\lambda\overline{P}_{n,r-1,1}(S),$$

$$(N-b+1\leq n\leq N \;;\; 2\leq r\leq k) \qquad \ldots(2.2.38.)$$

$$(S+K\lambda+\mu)\overline{P}_{n,1,1}(S) - k\lambda\overline{P}_{n-1,k,1}(S),$$

$$(N-b+1\leq n\leq N) \qquad \ldots(2.2.39.)$$

$$\begin{aligned}(S+k\lambda+\mu-k\lambda y-\mu z^{-b})\overline{P}_1(z,y,s) &= y+k\lambda y(z-y^k)Q_{k,1}(z,s)\\ &-\mu z^{-b}\sum_{n=0}^{a-1}\overline{P}_{n,1}(y,s)z^n+\mu z^{-b}\sum_{n=a}^{b}(z^b-z^n)\overline{P}_{n,1}(y,s)\\ &-k\lambda yz^{N+1}\overline{P}_{N,k,1}(S)+k\lambda y\overline{P}_{a-1,k,0}(S).\end{aligned}$$

$$\ldots(2.2.40.)$$

Where $\overline{Q}_{k,1}(z,s) = \sum_{n=0}^{N}\overline{P}_{n,k,1}(S)z^n$.

and $P_1(z,y,0) = y$.

now making the substitution y= = $\left(\frac{1}{k\lambda}\right)(S+\mu+k-\mu z^{-b})$ in equation (2.2.40.), we get after simplification;

$$\left[z^{bk+1} - \left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{k} \right] Q_{k,1}(z,s)$$

$$= -\frac{z^{bk}}{k\lambda} + \frac{\mu}{k\lambda}\sum_{n=0}^{a-1}\sum_{r=1}^{k}\overline{P}_{n,r,1}(S)z^{n+bk-br}\left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{r-1}$$

$$-\frac{\mu}{k\lambda}\sum_{n=a}^{b-1}\sum_{r=1}^{k}\overline{P}_{n,r,1}(S)(z^{b}-z^{n})z^{bk-br}\left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{r-1}$$

$$+z^{bk+N+1}\overline{P}_{N,k,1}(S)+z^{bk}\,\overline{P}_{a-1,k,0}(S). \qquad \ldots(2.2.41.)$$

Now to obtain the value of $\overline{P}_{a-1,k,0}(t)$ we have from (2.2.33.)

$$\overline{P}_{n,r,0}(S) = A\overline{P}_{n,r-1,0}(S)+B\overline{P}_{n,r,1}(S).$$

$$(0\le n\le a-1 \ ; \ 2\le r\le k) \qquad \ldots(2.2.42.)$$

where $A = \frac{k\lambda}{S+k\lambda}$ & $B = \frac{\mu}{S+k\lambda}$

Put r =k,k-1, ... 2 in (2.2.42.), we get

$$\overline{P}_{n,k,0}(S) = A\,\overline{P}_{n,k-1,0}(S)+B\overline{P}_{n,k,1}(S),$$

$$\bar{P}_{n,k-1,0}(S) = A\,\bar{P}_{n,k-2,0}(S)+B\bar{P}_{n,k-1,1}(S),$$

$$\begin{matrix} . & . & . \\ . & . & . \end{matrix}$$

$$\bar{P}_{n,2,0}(S) = A\bar{P}_{n,1,0}(S)+B\bar{P}_{n,2,0}(S).$$

Adding the above relations after multiplying by $1, A, A^2 \ldots$ respectively we have

$$\bar{P}_{n,k,0}(S) = A^{k-1}\bar{P}_{n,1,0}(S)+B\sum_{r=2}^{k} \bar{P}_{n,r,1}(S)A^{k-r}. \quad \ldots(2.2.43.)$$

Now consider (2.2.32.)

$$(S+k\,\lambda)\bar{P}_{n,1,0}(S) = k\,\lambda\bar{P}_{n-1,k,0}(S)+\mu\bar{P}_{n,1,1}(S).$$

$$(1 \le n \le a-1)$$

From the above relation we get

$$\bar{P}_{n,k,0}(S) = A^{k}\,\bar{P}_{n-1,k,0}(S)+B\sum_{r=1}^{k} \bar{P}_{n,r,1}(S)A^{k-r}. \quad \ldots(2.2.44.)$$

In the above relation putting successively $n=a-1, a-2, \ldots 1$, we get

$$\bar{P}_{a-1,k,0}(S) = A^{ak-k}\bar{P}_{0,k,0}(S)+B\sum_{r=1}^{k} A^{k-r} \sum_{n=1}^{a-1} A^{k(a-1-n)}\bar{P}_{n,r,1}(S).$$

$$\ldots(2.2.45.)$$

Again consider (2.2.33.)

$$\bar{P}_{n,r,0}(S) = A\bar{P}_{n,r-1,0}(S)+B\bar{P}_{n,r,1}(S),$$

In the above relation putting successively $r=k,k-1$ ....,2 and multiplying the equations by $1,A,A^2$, ... A^{k-2} respectively and then adding we get :

$$\bar{P}_{n,k,0}(S) = A^{k-1}\ \bar{P}_{n,1,0}(S)+B\sum_{r=2}^{k} A^{k-r}\ \bar{P}_{n,r,1}(S) \quad \ldots(2.2.46.)$$

Put $n=0$ in (2.2.46.), we have

$$\bar{P}_{0,k,0}(S)= A^{k-1}\bar{P}_{0,1,0}(S)+B\sum_{r=2}^{k} A^{k-r}\ \bar{P}_{0,r,1}(S). \quad \ldots(2.2.47.)$$

Now from (2.2.31.) we get

$$\bar{P}_{0,1,0}(S) = B\ \bar{Q}_{0,1,1}(S). \quad \ldots(2.2.48.)$$

Putting the value of $\bar{P}_{0,1,0}(S)$ from (2.2.48.) in (2.2.47.), we get

$$\bar{P}_{0,k,0}(S) = B\sum_{r=1}^{k-r} A^{k}\ \bar{P}_{0,r,1}(S). \quad \ldots(2.2.49.)$$

Putting the value of $\bar{P}_{0,k,0}(S)$ from (2.2.49.) in (2.2.45.)we get

$$\bar{P}_{a-1,k,0}(S) = \frac{\mu}{k\lambda}\sum_{r=1}^{k}\sum_{n=0}^{a-1}\left[\frac{k\lambda}{S+k\lambda}\right]^{k(a-n)-r}\bar{P}_{n,r,1}(S).$$

$$\ldots(2.2.50.)$$

Putting the value of $\bar{P}_{a-1,k,0}(S)$ from (2.2.50.) in (2.2.41.), we get

$$\left[z^{bk+1} - \left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{k}\right]Q_{k,1}(z,s)$$

$$= -\frac{z^{bk}}{k\lambda} + \frac{\mu}{k\lambda}\sum_{n=0}^{a-1}\sum_{r=1}^{k}\bar{P}_{0,r,1}(z)z^{n+bk-br}\left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{r-1}$$

$$-\frac{\mu}{k\lambda}\sum_{n=a}^{b-1}\sum_{r=1}^{k}\bar{P}_{n,r,1}(s)(z^{b}-z^{n})z^{bk-br}\left[\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right]^{r-1}$$

$$+ z^{bk}\left(\frac{\mu}{s+k\lambda}\right)\sum_{n=0}^{a-1}\sum_{r=1}^{k}\left(\frac{k\lambda}{s+k\lambda}\right)^{k(a-n)-r}\bar{P}_{n,r,1}(s)$$

$$+ z^{bk+N+1}\bar{P}_{N,k,1}(s). \qquad \ldots(2.2.51)$$

The characteristic equation of (2.2.51) is given by

$$z - \left[\frac{(s+\mu+k\lambda)-\mu z^{-b}}{k\lambda}\right]^{k} = 0. \qquad \ldots(2.2.52)$$

Application of Rouche's theorem to (2.2.52) yields that if $\rho=(\lambda/b\mu)<1$, there is exactly one root outside, and at least $(bk-1)$ roots are inside the unit circle $|z|=1$ (if $s=0$, then exactly $(bk-1)$ roots are

inside, one root outside and one root 1.) moreover the single root $\alpha(S)$ lying outside the unit circle is real. The equation (2.2.52) has no dependence on the quorum size 'a'.

using the standard analytic function theory arguments it follows from (2.2.51) and (2.2.52) that

$$Q_{k,1}(z) = \sum_{n=0}^{N} \frac{B(s)\ z^{n}}{\alpha^{n+1}(s)} ,$$

Which in view of

$$\bar{P}_{n,k,1}(s) = \frac{\bar{P}_{0,k,1}(s)}{\alpha^{n}} . \qquad \ldots(2.2.53.)$$

$$(0 \le n \le N)$$

To obtain the value of $\bar{P}_{n,r,1}(S)$, $1 \le n \le N-b$, $2 \le r \le k$ consider (2.2.37)

$$(s + \mu + k\ \lambda)\bar{P}_{n,r,1}(s) = \mu\bar{P}_{n+b,r,1}(s) + k\ \lambda\bar{P}_{n,r-1,1}(s),$$

By the usual method we have from above equation

$$\bar{P}_{n,r,1}(s) = \left[\frac{(s+k\ \lambda+\mu-\ \mu z^{-b}}{k\ \lambda}\right]^{k-r} \bar{P}_{n,k,1}(s). \qquad \ldots(2.2.54.)$$

$$(1 \le n \le N-b\ ;\ 2 \le r \le k)$$

Putting the value of $\bar{P}_{n,k,1}(s)$ in the above relation, we get

$$\bar{P}_{n,r,1}(s) = \left[\frac{(s+k\lambda+\mu-\mu z^{-b})}{k\lambda}\right]^{k-r} \frac{\bar{P}_{0,k,1}(s)}{\alpha^{n}(s)}. \quad \ldots(2.2.55.)$$

$$(1 \le n \le N-b \; ; \; 2 \le r \le k)$$

From (2.2.38.), we obtain

$$\bar{P}_{n,r,1}(s) = \left(\frac{s+k\lambda+\mu}{k\lambda}\right)^{k-r+1} \frac{\bar{P}_{0,k,1}(s)}{\alpha^{n}}. \quad \ldots(2.2.56.)$$

$$(N-b+1 \le n \le N \; ; \; 2 \le r \le k)$$

Above result holds for r=1, hence we have

$$\bar{P}_{n,1,1}(s) = \left(\frac{s+k\lambda+\mu}{k\lambda}\right)^{k} \frac{\bar{P}_{0,k,1}(s)}{\alpha^{n}}. \quad \ldots(2.2.57.)$$

To obtain the value of $\bar{P}_{0,r,1}(s)$, consider (2.2.35.)

$$(s+k\lambda+\mu)\bar{P}_{0,r,1}(s)-k\lambda\bar{P}_{0,r-1,1}(s)-\mu\sum_{j=a}^{b}\bar{P}_{j,r,1}(s) = 0,$$

$$(2 \le r \le k)$$

or

$$\bar{P}_{0,r-1,1}(s) = -\frac{\mu}{k\lambda}\bar{P}_{0,k,1}(s)\left(\frac{s+k\lambda+\mu-\mu\alpha^{-b}}{k\lambda}\right)^{-r}\left[\frac{\alpha^{b+1}-\alpha^{a}}{\alpha^{a+b-1}(\alpha-1)}\right]$$

$$+\left(\frac{s+k\lambda+\mu}{k\lambda}\right)\bar{P}_{0,r,1}(s). \quad \ldots(2.2.58.)$$

In (2.2.58.) put r=r+1, r+2, ... k, successively then multiply the corresponding equations by $\left(\frac{s+k\lambda+\mu}{k\lambda}\right)^0$,

$\left(\frac{s+k\lambda+\mu}{k\lambda}\right)^1$, $\left(\frac{s+k\lambda+\mu}{k\lambda}\right)^2$, ... $\left(\frac{s+k\lambda+\mu}{k\lambda}\right)^{k-r+1}$, then adding all the equations, we get

$$\bar{P}_{0,r,1}(s) = \left(\frac{s+k\lambda+\mu}{k\lambda}\right)^{k-r} \bar{P}_{0,k,1}(s)$$

$$- E\left[F^{-r-1} + \left(\frac{s+k\lambda+\mu}{k\lambda}\right) F^{-r-2} + \ldots + \left(\frac{s+k\lambda+\mu}{k\lambda}\right)^{k-r-1} F^{-k}\right]. \quad \ldots(2.2.59.)$$

Where $E = \frac{\mu}{k\lambda} \bar{P}_{0,k,1}(S) \frac{(\alpha^{b+1} - \alpha^{a})}{\alpha^{a+b-1}(\alpha-1)}$

$$F = \left(\frac{s+k\lambda+\mu-\mu\alpha^{-b}}{k\lambda}\right)$$

The value of the second term in (2.2.59.) is obtained as

$$-E\left[F^{-r-1} + \left(\frac{S+k\lambda+\mu}{k\lambda}\right)F^{-r-2} + \ldots\ldots\ldots + \left(\frac{S+k\lambda+\mu}{k\lambda}\right)^{k-r-1} F^{-k}\right]$$

$$= -E\left[\frac{\alpha F^{-r} - \left(\frac{S+k\lambda+\mu}{k\lambda}\right)^{k-r}}{\alpha\,(-\mu\alpha^{-b}/k\lambda}\right]. \quad \ldots(2.2.60.)$$

Putting the value from (2.2.60.) in equation (2.2.59.) we get

$$\bar{P}_{0,r,1}(S) = \frac{\bar{P}_{0,k,1}(S)}{\alpha^{a-1}(S)(\alpha(S)-1)}\left[\left(\frac{k\lambda+\mu+S}{k\lambda}\right)^{k-r}\{\alpha^{a}(S)-\alpha^{b}(S)\}\right.$$

$$\left.+\{\alpha^{b+1}(S)-\alpha^{a}(S)\}\left(\frac{k\lambda+\mu+S-\mu\alpha^{-b}(S)}{k\lambda}\right)^{-r}\right]. \quad \ldots(2.2.61.)$$

$$(2\le r\le k)$$

To determine the value of $\bar{P}_{0,k,1}(S)$, we substitute

$$\bar{Q}_{k,1}(z,s) = \frac{\alpha(s)\bar{P}_{0,k,1}(s)(\alpha^{N+1}-z^{N+1})}{(\alpha(s)-Z)(\alpha^{N+1})}.$$

in equation (2.2.51), cross multiply by $\{\alpha(s)-z\}$, we get

$$\left[z^{bk+1}-\left(\frac{(s+\mu+k\lambda)z^{b}-\mu}{k\lambda}\right)^{k}\right]\frac{\alpha\bar{P}_{0,k,1}(s)(\alpha^{N+1}-z^{N+1})}{\alpha^{N+1}}$$

$$= -\frac{z^{bk}}{k\lambda}(\alpha-z)+\frac{\mu}{k\lambda}(\alpha-z)\sum_{n=0}^{a-1}\sum_{r=1}^{k}\bar{P}_{n,r,1}(s)z^{n+bk+br}.$$

$$\left[\frac{(s+k\lambda+\mu)z^{b}-\mu}{k\lambda}\right]^{r-1}-\frac{\mu}{k\lambda}(\alpha-z)\sum_{n=a}^{b-1}\sum_{r=1}^{k}\bar{P}_{n,r,1}(s)(z^{b}-z^{n}).$$

$$z^{bk-br}\left[\frac{(s+k\lambda+\mu)z^{b}-\mu}{k\lambda}\right]^{r-1}+z^{bk}(\alpha-z)\left(\frac{\mu}{s+k\lambda}\right)\sum_{n=0}^{a-1}\sum_{r=1}^{k}\left[\frac{k\lambda}{s+k\lambda}\right]^{k(a-n)-r}.$$

$$\bar{P}_{n,r,1}(s)+z^{bk+N+1}(\alpha-z)\bar{P}_{N,k,1}(s). \quad \ldots(2.2.62.)$$

Compare the coefficients of z^{bk+1} on both the sides of (2.2.62.), we get

$$\alpha(S)\bar{P}_{0,k,1}(S) = \frac{1}{k\lambda} + \bar{P}_{0,k,1}(S)\frac{\mu}{k\lambda}\sum_{n=a}^{b-1}\frac{1}{\alpha^{n-1}(S)}\sum_{r=1}^{k}\left(\frac{S+k\lambda+\mu}{k\lambda}\right)^{r-1}.$$

$$\left[\frac{S+k\lambda+\mu-\mu Z^{-b}}{k\lambda}\right]^{-r} - \frac{\mu}{(S+k\lambda)}\bar{P}_{0,k,1}(S)\sum_{n=0}^{a-1}\frac{1}{\alpha^{n-1}(S)}.$$

$$\sum_{r=1}^{k}\left[\frac{k\lambda}{S+k\lambda}\right]^{k(a-n)-r}\left[\frac{S+k\lambda+\mu+\mu Z^{-b}}{k\lambda}\right]^{-r}. \qquad \ldots(2.2.63.)$$

The values of second and third terms in the R.H.S. of (2.2.63.) are obtained as follows

$$\bar{P}_{0.k.1}(s)\frac{\mu}{k\lambda}\sum_{n=a}^{b-1}\frac{1}{\alpha^{n-1}(s)}\sum_{r=1}^{k}\left[\frac{S+k\lambda+\mu}{k\lambda}\right]^{r-1}\kappa\left(\frac{S+k\lambda+\mu-\mu Z^{-b}}{k\lambda}\right)^{-r}$$

$$= -\frac{\bar{P}_{0,k,1}(S)}{\alpha^{a-1}(S)\{\alpha(S)-1\}}\,[\alpha^{b}(S)-\alpha^{a}(S)]\,\left[\alpha-\left(\frac{S+k\lambda+\mu}{k\lambda}\right)^{k}\right].$$

$$\ldots(2.2.64.)$$

and

$$\bar{P}_{0,k,1}(s)\frac{\mu}{(s+k\lambda)}\sum_{n=0}^{a-1}\frac{1}{\alpha^{n-1}(s)}\sum_{r=1}^{k}\left[\frac{k\lambda}{s+k\lambda}\right]^{k(a-n)-r}.$$

$$\left(\frac{s+k\lambda+\mu-\mu z^{-b}}{k\lambda}\right)^{-r}$$

$$= \frac{\bar{P}_{0,k,1}(s)k\lambda\,\{\alpha^{a}(s)(k\lambda)^{ak}-(s+k\lambda)^{ak}\}}{(s+k\lambda)^{ak}\,\{1-\alpha^{-b}(s)\}\,\alpha^{a-2}(s)}. \qquad \ldots(2.2.65.)$$

Putting the values from (2.2.64.) and (2.2.65.) in (2.2.63.), we get after simplification

$$\bar{P}_{0,k,1}(S) = \frac{\alpha^{a-1}(S)\{\alpha(S)-1\}}{k\lambda}$$

$$\left[\begin{array}{l} \{\alpha^{b+1}(S)-\alpha^{a}(S)\}-\{\alpha^{b}(S)-\alpha^{a}(S)\}\{\frac{S+k\lambda+\mu}{k\lambda}\}^{k} \\ \\ +\frac{\alpha(S)k\lambda\{\alpha(S)-1\}}{(1-\alpha^{-b})}\{\alpha^{a}(S)(\frac{k\lambda}{S+k\lambda})^{ak}-1\} \end{array}\right]^{-1}.$$

...(2.2.66.)

2.3 BUSY PERIOD DISTRIBUTION

The probability density function (p.d.f.) of the busy period is given by

$$f(t) = \sum_{n=0}^{a-1} \sum_{r=1}^{k} \frac{d}{dt} P_{n,r,0}(t). \qquad \ldots(2.3.1.)$$

The Laplace transform of the busy period p.d.f. is

$$\bar{f}(S) = S\sum_{n=0}^{a-1} \sum_{r=1}^{k} \bar{P}_{n,r,0}(S) = 1-S\sum_{n=0}^{N} \sum_{r=1}^{k} \bar{P}_{n,r,1}(S)=1-S\bar{P}_{1,1,S}.$$

$$\ldots(2.3.2.)$$

Substituting Y=Z=1 in equation (2.2.40), we get

$$S\bar{P}_{1(1,1,S)}=1-\mu\sum_{n=0}^{a-1} \bar{P}_{n,1}(1,s)-k\ \lambda\bar{P}_{N,k,1}(S)+k\ \lambda\bar{P}_{a-1,k,0}(S).$$

With above value the right side of (2.3.2) reduces to

$$\bar{f}(S)=\mu\sum_{n=0}^{a-1} \bar{P}_{n,1}(1,S)+k\ \lambda\bar{P}_{N,k,1}(S)+k\ \lambda\ \bar{P}_{a-1,k,0}(S).$$

$$\ldots(2.3.3.)$$

To obtain the values of different terms in (2.3.3) we get

$$\sum_{n=0}^{a-1} \bar{P}_{n,1}(1,S) = \sum_{n=0}^{a-1}\sum_{r=1}^{k} \bar{P}_{n,r,1}(t)y^{r}=\sum_{r=1}^{k} \bar{P}_{0,r,1}(S)$$

$$+\sum_{n=1}^{a-1} \sum_{r=1}^{k} \bar{P}_{n,r,1}(S). \qquad \ldots(2.3.4.)$$

Now

$$\sum_{r=1}^{k} \bar{P}_{o,r,1}(S) = \frac{\bar{P}_{0,k,1}(S)}{\alpha^{a-1}(S)\{\alpha(S)-1\}} \cdot$$

$$\sum_{r=1}^{k}\left[\begin{array}{l} \left(\frac{S+k\lambda+u}{k\lambda}\right)^{k-r}\{\alpha^{a}(S)-\alpha^{b}(S)\} \\ +\{\alpha^{b+1}(S)-\alpha^{a}(S)\}\left\{\frac{S+k\lambda+\mu-\mu\alpha^{-b}(S)}{k\lambda}\right\}^{-r} \end{array}\right]$$

$$= \frac{\mu\bar{P}_{0,k,1}(S)}{\alpha^{a-1}(S)\{\alpha(S)-1\}}\left[\begin{array}{l} \left[\frac{(S+k\lambda+\mu)^{k}-(k\lambda)^{k}}{(\mu+S)(k\lambda)^{k-1}}\right]\{\alpha^{a}(S)-\alpha^{b}(S)\} \\ \\ +\{\alpha^{b+1}(S)-\alpha^{a}(S)\}\left[\frac{k\lambda\ (\alpha(S)-1)}{\alpha(s)\{S+\mu-\mu\alpha^{-b}(S)\}}\right] \end{array}\right].$$

...(2.2.5.)

Now consider

$$\sum_{n=1}^{a-1}\sum_{r=1}^{k}\bar{P}_{n,r,1}(s)=\bar{P}_{0,k,1}(s)\sum_{r=1}^{k}\left[\frac{k\lambda}{s+k\lambda+\mu-\mu\alpha^{-b}}\right]^{-r}\sum_{n=1}^{a-1}\frac{1}{\alpha^{n-1}}$$

we get after simplification

$$\sum_{n=1}^{a-1}\sum_{r=1}^{k}\bar{P}_{n,r,1}(S) = \frac{\bar{P}_{0,k,1}(S)k\lambda\mu}{\alpha^{a-1}(S)\{\alpha(S)-1\}} \cdot$$

$$\left[\begin{array}{l} \left[\frac{\left(\frac{S+k\lambda+\mu}{k\lambda}\right)^{k}-1}{(\mu+S)}\right]\left[\alpha^{a}(S)-\alpha^{b}(S)\right] \\ +\frac{\{\alpha^{b+1}(S)-\alpha^{a}(S)\}\{\alpha(S)-1}{\alpha(S)\{S+\mu-\mu\alpha^{-b}(S)\}} \\ \\ +\frac{\{\alpha^{a-1}(S)-1\}\{\alpha(S)-1\}}{\{S+\mu-\mu\alpha^{-b}(S)\}} \end{array}\right],$$

...(2.3.6.)

$$k\lambda\bar{P}_{N,k,1}=\frac{k\lambda\bar{P}_{0,k,1}(S)}{\alpha^{N}}\,, \qquad \ldots(2.3.7.)$$

$$k\lambda\bar{P}_{a-1,k,0}(S)=\frac{\bar{P}_{0,k,1}(S)k\lambda\{\alpha^{a}(S)(k\lambda)^{ak}-(S+k\lambda)^{ak}\}}{(S+k\lambda)^{ak}\{1-\alpha^{-b}(S)\}\alpha^{a-2}(S)}\,.$$

$$\ldots(2.3.8.)$$

Now putting the values from (2.3.4)-(2.3.8) in (2.3.3),we get after simplification

$$\bar{f}(S)=\mu\,\frac{\left[\dfrac{\{\alpha(S)-1\}\{\alpha^{b}(S)-1\}}{\{S+\mu-\mu\alpha^{-b}(S)\}}+\dfrac{\{\alpha^{b}(S)-\alpha^{a}(S)\}}{(\mu+S)}\{1-(\dfrac{S+\mu+k\lambda}{k\lambda})^{k}\}+\dfrac{\{\alpha^{a}(S)(k\lambda)^{ak}-(S+k\lambda)^{ak}\}}{(S+k\lambda)^{ak}\{1-\alpha^{-b}(S)\}\alpha^{a-2}(S)}+\dfrac{1}{\alpha^{N}\mu}\right]}{\left[\{\alpha^{b+1}(S)-\alpha^{a}(S)\}-\{\alpha^{b}(S)-\alpha^{a}(S)\}(\dfrac{S+k\lambda+\mu}{k\lambda})^{k}+\dfrac{\alpha(S)k\lambda\{\alpha(S)-1\}}{1-\alpha^{-b}(S)}\{\alpha^{a}(S)(\dfrac{k\lambda}{S+k\lambda})^{ak}-1\}\,.\right]}\,.$$

$$\ldots(2.3.9.)$$

2.4 COMPUTER PROGRAMME

```
#include<stio.h>
#include<math.h>
main()
{
  int i=0,temp,b,k,al;
  float zn[6],tmp;
  float alpha,bita,first_fact,second_fact,third_fact;
  float cal_fact_1,cal_fact_2,fun_z;
  float a,pw,fun_z_dn,first_fact_dn,second_fact_dn;
  float third_fact_dn,final_fun;
  clrscr();
for(k=1; k<=20; k++)
{
  clrscr();
  printf("\n\t\t\t\tk = %d\n\n", k);
  printf("------------------------------------"\
      "-------------------------------");
  printf("\nalpha   :b-       1       2       3      4"\
          "         5");
  printf("\n-----------------------------------------"\
          "--------------------------\n");
/* getch();*/
 alpha=0.05;
 for(al=1;  al<= 10;  al++)
 {
   i=0;
   zn[0] =1.25;
/* printf("IN %d LOOP FOR ALPHA=%f\n",al,alpha);*/
 for(b=1; b<5; b+=1)
  {
   pw = (b*k + 1.0);
```

```
        first_fact = pow(zn[i],pw);
        bita         = alpha / k;
        second_fact= (1+bita) * pow(zn[i],b;
        third_fact = bita;
        cal_fact_1 = second_fact - third_fact;
        cal_fact_2 + pow(cal_fact_1, k);
        fun_z        = first_fact _ cal_fact_2;
        first_fact_dn = pw * pow(zn[i], b*k);
        second_fact_dn = k * pow(cal_fact_1, k-1);
        third_fact_dn = b * (1.0+bita) * pow (zn[i],b-1);
        fun_z_dn = first_fact_dn - (second_fact_dn * third_fact_
        final_fun = fun_z / fun_z_dn;
        tmp = zn[i]; i++;
        zn[i]   = tmp - final_fun;
    }
    printf("%.2f", alpha);
    for (temp = 0; temp<5; temp++)
      {
       printf("      %f", zn[temp]);
      }
    printf("\n");
      alpha += .05;
    }
    /* getch();*/
     printf("----------------------------------------"\
           "-------------------------------------\n");
    getch();
    }
    }
```

SAMPLE LISTINGS OF THE ROOT α OF EQN(1.2.52)

			K=2			
RHO	b =	1	2	3	4	5
0.050000		1.250000	1.072477	1.015859	1.001427	1.000017
0.100000		1.250000	1.073588	1.016803	1.001694	1.000027
0.150000		1.250000	1.074769	1.017905	1.002074	1.000048
0.200000		1.250000	1.076029	1.019221	1.002666	1.000114
0.250000		1.250000	1.077381	1.020836	1.003739	1.001899
0.300000		1.250000	1.078838	1.022890	1.006398	0.999736
0.350000		1.250000	1.080416	1.025629	1.028830	0.999207
0.400000		1.250000	1.082137	1.029543	0.991561	1.000189
0.450000		1.250000	1.084025	1.035758	0.997610	1.000020
0.500000		1.250000	1.086111	1.047583	1.000273	1.000000

CHAPTER 3

3.1 INTRODUCTION

In chapter 2 the transient state of the queueing system $E_k/M^{a,b}/1/N$ was studied under the assumption of finite waiting space. In the present chapter we consider a particular case i.e. the steady state of the system which was discussed in chapter 1 . The steady state of the present queueing system $E_k/M^{a,b}/1/N$ with finite waiting space is a modification of the system studied by Easton and chaudhry [42] with infinite waiting space. The proposed system also obeys the same general bulk service rule. Similar models were earlie: discussed by Borthakur [23] and Medhi [133] Neuts [146] One can also refer the work of Holman [90].

Purpose of the present study is to extend the results of Easton and Chaudhry [42] when the waiting space is finite, say N. All the analytic results are expresed in terms of the unique root (of the underlying characteristic equation), which is easy to compute using one of the several available numerical methods. Here we assume that the interarrival times have Erlang distribution with a mean for each phase of $1/k\lambda$ such that the overall mean interval time is $1/\lambda$. In many industrial processes units may go through several exponential phases before they are out and the processes have limited waiting space. However it may be stated that the phases need not have physical meaning they are introduced to generalize the input process.

In modeling real life transportation systems, we often find that bulk service queueing structures occupy an important place. Taxi stand, unscheduled car ferry, a

single ground floor station of an elevator are some of the important practical situations wherein the model under investigation has applications. Since bulk service queueing systems are difficult to analyse, some of the recent researchers have made many simplifying assumptions in order to obtain limited results for such systems. These assumptions are poisson processes for the input and exponential service. But these assumptions may not hold in many practical situations. In the present chapter interarrival times follow an Erlangian distribution. Further more the results are presented in such a way that computations are easily done, thus making the model potentially useful.

In 3.2 of this chapter the steady state random point results for single server bulk service system $E_k/M^{a,b}/1/N$ are discussed. In 3.3 the imbedded Markov results using the Erlangian technique and steady state results, in 3.4 the expected number in queue at random and arrival epochs are discussed and the sample listings for the bulk service queue $E_k/M^{a,b}/1/N$ are also presented in tabulor form.

3.2 THE RANDOM POINT RESULTS

The present queueing system $E_k/M^{a,b}/1/N$ with finite waiting space is defined as follows :
Service is governed by a general rule : Service begins only when "a" units (quorum) are present. If the queue length is $\geq$ a but $\leq$ b the entire queue is taken up : if there are more than b units, then the server accepts the first b units. We define:

$Q_{n,r}(t)$ = Probability that at time 't', there are n units waiting in queue the arrival is in phase r and the server is busy.

$0 \leq n \leq N$

$1 \leq r \leq k$

$P_{n,r}(t)$ refers to the situation when server is idle ;

$0 \leq n < a$

$1 \leq r \leq k$

$Q_{n,r} = \lim_{t\to\infty} Q_{n,r}(t)$.

$P_{n,r} = \lim_{t\to\infty} P_{n,r}(t)$.

Following keilson and kooharian (1960), we have the following transition equations for the system :

$$P_{0,1}(t+\Delta t) = (1-k\ \lambda\Delta t)P_{0,1}(t)+\mu\Delta t+Q_{0,1}(t), \qquad \ldots(3.2.1.)$$

$$P_{n,1}(t+\Delta t) = (1-k\ \lambda\Delta t)P_{n,1}(t)+\mu\Delta t(1-k\ \lambda\Delta t)Q_{n,1}(t)$$

$$+k\ \lambda\Delta tP_{n-1,k}(t), \qquad \ldots(3.2.2.)$$

$$(1\leq n\leq a-1)$$

$$P_{n,r}(t+\Delta t) = (1-k\ \lambda\Delta t)P_{n,r}(t)+\mu\Delta t((1-k\ \lambda\Delta t)Q_{n,r}(t)+$$

$$k\ \lambda\Delta tP_{n,r-1}(t), \qquad \ldots(3.2.3.)$$

$$(0\leq n\leq a-1;\ 2\leq r\leq k)$$

$$Q_{0,1}(t+\Delta t) = Q_{0,1}(t)(1-k\ \lambda\Delta t)(1-\mu\Delta t)+(1-k\ \lambda\Delta t)\mu\Delta t\sum_{j=a}^{b} Q_{j,1}(t)$$

$$+P_{a-1,k}(t)k\ \lambda\Delta t, \qquad \ldots(3.2.4.)$$

$$Q_{0,r}(t+\Delta t) = Q_{0,r}(t)(1-k\ \lambda\Delta t)(1-\mu\Delta t)+Q_{0,r-1}(t)k\ \lambda\Delta t(1-\mu\Delta t)$$

$$+(1-k\ \lambda\Delta t)\mu\Delta t\sum_{j=a}^{b} Q_{j,r}(t), \qquad \ldots(3.2.5.)$$

$$(2\leq r\leq k)$$

$$Q_{n,1}(t+\Delta t) = Q_{n,1}(t)(1-k\ \lambda\Delta t)(1-\mu\Delta t)+Q_{n+b,1}(\mu\Delta t)(1-k\ \lambda\Delta t)$$

$$+Q_{n-1,k}(k\ \lambda\Delta t)(1-\mu\Delta t), \qquad \ldots(3.2.6.)$$

$$(1\leq n\leq N-b)$$

$$Q_{n,r}(t+\Delta t) = Q_{n,r}(1-k\ \lambda\Delta t)(1-\mu\Delta t)+Q_{n+b,r}(t)\mu\Delta t(1-k\ \lambda\Delta t)$$

$$+Q_{n,r-1}(t)(k\ \lambda\Delta t)(1-\mu\Delta t), \qquad \ldots(3.2.7.)$$

$$(1\leq n\leq N-b\ ;\ 2\leq r\leq k)$$

$$Q_{n,r}(t+\Delta t) = Q_{n,r}(1-k\ \lambda\Delta t)(1-\mu\Delta t)+Q_{n,r-1}(t)(k\ \lambda\Delta t)(1-\mu\Delta t),$$

$$(N-b+1\leq n\leq N\ ;\ 2\leq r\leq k) \qquad \ldots(3.2.8.)$$

$$Q_{n,1}(t+\Delta t) = Q_{n,1}(t)(1-k\ \lambda\Delta t)(1-\mu\Delta t)+Q_{n-1,k}(k\ \lambda\Delta t)(1-\mu\Delta t).$$

$$(N-b+1\leq n\leq N) \qquad \ldots(3.2.9.)$$

Transposing and letting $\Delta t \to 0$, we have

$$\frac{d}{dt}P_{0,1}(t) = -k\lambda P_{0,1}(t)+\mu Q_{0,1}(t), \qquad \ldots(3.2.10.)$$

$$\frac{d}{dt}P_{n,1}(t) = -k\lambda P_{n,1}(t)+\mu Q_{n,1}(t)+k\lambda P_{n-1,k}(t), \qquad \ldots(3.2.11.)$$

$$(1\le n\le a-1)$$

$$\frac{d}{dt}P_{n,r} = \mu Q_{n,r}-k\lambda P_{n,r}+k\lambda P_{n,r-1}, \qquad \ldots(3.2.12.)$$

$$(2\le n\le a-1 \;;\; 2\le r\le k)$$

$$\frac{d}{dt}Q_{0,1} = -(k\lambda+\mu)Q_{0,1}+\mu\sum_{j=a}^{b} Q_{j,1} + k\lambda P_{a-1,k}, \qquad \ldots(3.2.13.)$$

$$\frac{d}{dt}Q_{0,r} = -(k\lambda+\mu)Q_{0,r}+k\lambda Q_{0,r-1}+\mu\sum_{j=a}^{b} Q_{j,r}, \qquad \ldots(3.2.14.)$$

$$(2\le r\le k)$$

$$\frac{d}{dt}Q_{n,1} = -(k\lambda+\mu)Q_{n,1}+\mu Q_{n+b,1}+k\lambda Q_{n-1,k}, \qquad \ldots(3.2.15.)$$

$$(1\le n\le N-b)$$

$$\frac{d}{dt}Q_{n,r} = -(k\lambda+\mu)Q_{n,r}+k\lambda Q_{n,r-1}+\mu Q_{n+b,r}, \qquad \ldots(3.2.16.)$$

$$(1\le n\le N-b \;;\; 2\le r\le k)$$

$$\frac{d}{dt}Q_{n,r} = -(k\lambda+\mu)Q_{n,r}+k\lambda Q_{n,r-1}, \qquad \ldots(3.2.17.)$$

$$(N-b+1\le n\le N \;;\; 2\le r\le k)$$

$$\frac{d}{dt}Q_{n,1} = -(k\lambda+\mu)Q_{n,1}+k\lambda Q_{n-1,k} \quad \ldots(3.2.18.)$$

$$(N-b+1\leq n\leq N)$$

In steady state the (3.2.10)-(3.2.18) reduces in the following form.

$$k\lambda P_{0,1} = \mu Q_{0,1} \quad , \quad \ldots(3.2.19.)$$

$$K\lambda P_{n,1} = \mu Q_{n,1} + k\lambda P_{n-1,k} \quad , \quad \ldots(3.2.20.)$$

$$(1\leq n\leq a-1)$$

$$K\lambda P_{n,r} = \mu Q_{n,r} + K\lambda P_{n,r-1} \quad , \quad \ldots(3.2.21.)$$

$$(0 \leq n \leq a-1 \; ; \; 2\leq r\leq k)$$

$$(K\lambda+\mu)\, Q_{0,1} = K\lambda P_{a-1,K} + \mu \sum_{j=a}^{b} Q_{j,1} \quad , \quad \ldots(3.2.22.)$$

$$(K\lambda+\mu)Q_{0,r} = K\lambda Q_{0,r-1} + \mu \sum_{j=a}^{b} Q_{j,r} \quad , \quad \ldots(3.2.23.)$$

$$(2\leq r \leq k)$$

$$(K\lambda+\mu)\, Q_{n,1} = K\lambda Q_{n-1,k} + \mu Q_{n+b,1} \quad , \quad \ldots(3.2.24.)$$

$$(1\leq n \leq N-b)$$

$$(K\lambda+\mu)\, Q_{n,r} = K\lambda Q_{n,r-1} + \mu Q_{n+b,r} \quad , \quad \ldots(3.2.25.)$$

$$(1\leq n \leq N-b \; ; \; 2\leq r \leq k)$$

$$(K\lambda + \mu)\, Q_{n,r} = K\lambda\, Q_{n,r-1}\,, \qquad \ldots(3.2.26.)$$

$$(N - b+1 \le n \le N; 2 \le r \le k)$$

$$(K\lambda + \mu)\, Q_{n,1} = K\lambda\, Q_{n-1,k}\,. \qquad \ldots(3.2.27.)$$

$$(N - b+1 \le n \le N)$$

Note :(3.2.21), (3.2.23) ,(3.2.27) and (3.2.28) will not occure if k=1 and fails to hold if a=1.

Define the following generating functions

$$Q_r(z) = \sum_{n=0}^{N} Q_{n,r}\, z^n \;; \quad |z| < 1\,, \qquad \ldots(3.2.28.)$$

$$Q(x,z) = \sum_{r=1}^{k} Q_r\,(z)\, x^r\,. \qquad \ldots(3.2.29.)$$

Following the usual technique of generating function analysis from equations

$$Q(x,y) = \sum_{r=1}^{k} Q_r(z)x^r$$

$$= \sum_{r=1}^{k} x^r \sum_{n=0}^{N} Q_{n,r} Z^n\,.$$

$$Q(x,z) = \sum_{r=1}^{k} Q_{0,r}x^r + \sum_{n=1}^{N-b} z^n \Big(\sum_{r=1}^{k} Q_{n,r}x^r \Big) +$$

$$\sum_{n=N-b+1}^{N} z^n \sum_{r=1}^{k} Q_{n,r}x^r\,. \qquad \ldots(3.2.30.)$$

Now taking appropriate summations of each term of (3.2.30) we get

$$\sum_{r=1}^{k} Q_{0,r}\kappa^{r} = \frac{k\lambda\kappa}{(k\lambda+\mu)}P_{a-1,k}+\frac{\mu}{k\lambda+\mu}\sum_{j=a}^{b}\sum_{r=1}^{k} Q_{j,r}\kappa^{r}+$$

$$\frac{k\lambda}{(k\lambda+\mu)}\sum_{r=2}^{k} Q_{0,r-1}\kappa^{r}, \qquad \ldots(3.2.31.)$$

$$\sum_{n=1}^{N-b} z^{n}(\sum_{r=1}^{k} Q_{n,r}\kappa^{r}) = \sum_{n=1}^{N-b} z^{n}\left[\frac{k\lambda\kappa}{(k\lambda+\mu)}Q_{n-1,k}+\frac{\mu}{(k\lambda+\mu)}\sum_{r=1}^{k} Q_{n+b,r}\kappa^{r}\right.$$

$$\left.+\frac{k\lambda}{(k\lambda+\mu)}\sum_{r=2}^{k} Q_{n,r-1}\kappa^{r}\right], \qquad \ldots(3.2.32.)$$

$$\sum_{n=N-b+1}^{N} z^{n}(\sum_{r=1}^{k} Q_{n,r}\kappa^{r}) = \sum_{n=N-b+1}^{N} z^{n}\left[\frac{k\lambda\kappa}{(k\lambda+\mu)}Q_{n-1,k}+\right.$$

$$\left.\frac{k\lambda}{(k\lambda+\mu)}\sum_{r=2}^{k} Q_{n,r-1}\kappa^{r}\right]. \qquad \ldots(3.2.33.)$$

Putting the values from (3.2.31)-(3.2.33) in (3.2.30), we get

$$(k\lambda+\mu)Q(\kappa,z) = k\lambda\kappa P_{a-1,k}+k\lambda\sum_{n=0}^{N}\sum_{r=2}^{k} Q_{n,r-1}\kappa^{r}+\mu\sum_{j=a}^{b}\sum_{r=1}^{k} Q_{j,r}\kappa^{r}$$

$$+\mu\sum_{n=1}^{N-b} z^{n}\sum_{r=1}^{k} Q_{n+b,r}\kappa^{r}+$$

$$k\lambda\kappa\sum_{n=1}^{N} Q_{n-1,k}z^{n}. \qquad \ldots(3.2.34.)$$

To obtain the values of certain terms in (3.2.34) by usual methods.

consider (3.2.21) i.e.

$$k\ \lambda P_{n,r} = \mu Q_{n,r} + k\ \lambda P_{n,r-1}\ . \qquad \ldots(3.2.35.)$$

$$(0 \le n \le a-1\ ;\ 2 \le r \le k)$$

Put $n=a-1$ in above equation, we get

$$k\ \lambda P_{a-1,r} = \mu Q_{a-1,r} + k\ \lambda P_{a-1,r-1}\ . \qquad \ldots(3.2.36.)$$

$$(2 \le r \le k)$$

Putting $r=k, k-1 \ldots 2$ in (3.2.36) and solving recursively, we get

$$k\ \lambda P_{a-1,k} = \mu \sum_{r=2}^{k} Q_{a-1,r} + k\ \lambda P_{a-1,1}\ . \qquad \ldots(3.2.37.)$$

By generalizing the above rsulst for $n<a$, we get

$$k\ \lambda P_{n,k} = \mu \sum_{r=2}^{k} Q_{n,r} + k\ \lambda P_{n,1}\ . \qquad \ldots(3.2.38.)$$

From (3.2.20), above equation reduces

$$k\ \lambda P_{n,k} = \mu \sum_{r=1}^{k} Q_{n,r} + k\ \lambda P_{n-1,k}\ . \qquad \ldots(3.2.39.)$$

Again putting $n=a-1,\ a-2 \ldots\ldots 1$ in (3.2.39) and solving recursively, we get

$$k\ \lambda P_{a-1,k} = \mu \sum_{n=1}^{a-1} \sum_{r=1}^{k} Q_{n,r} + k\ \lambda P_{0,k}\ . \qquad \ldots(3.2.40.)$$

Again from (3.2.21) put $r=k$, $k-1$, $k-2,\ldots 2$ and solving recursively, we get

$$k\,\lambda P_{n,k} = \mu\sum_{r=2}^{k} Q_{n,r} + k\,\lambda P_{n,1}\ . \qquad \ldots(3.2.41.)$$

we put $n=0$ in (3.2.40) to get

$$k\,\lambda P_{0,k} = \mu\sum_{r=1}^{k} Q_{0,r}\ . \qquad \ldots(3.2.42.)$$

On putting the value of $k\,\lambda P_{0,k}$ from (3.2.42) in(3.2.40), we get

$$k\,\lambda P_{a-1,k} = \mu\sum_{n=0}^{a-1}\sum_{r=1}^{k} Q_{n,r}\ . \qquad \ldots(3.2.43.)$$

$$k\,\lambda\sum_{n=0}^{N}\sum_{r=2}^{k} Q_{n,r-1}\kappa^{r} = k\,\lambda\kappa Q(\kappa,z) - k\,\lambda\,\kappa^{k+1}Q_{k}(z)\ , \qquad \ldots(3.2.44.)$$

$$\mu\sum_{n=1}^{N-b} z^{n}\sum_{r=1}^{k} Q_{n+b,r}\kappa^{r} = \mu z^{-b}Q(\kappa,z) - \mu z^{-b}\sum_{r=1}^{k}\kappa^{r}\sum_{n=0}^{b} Q_{n,r}z^{n}\ ,$$

$$\ldots(3.2.45.)$$

$$k\,\lambda\kappa\sum_{n=1}^{N} Q_{n-1,k}z^{n} = k\,\lambda\kappa z Q_{k}(z) - k\,\lambda\kappa z^{N+1}Q_{N,k}\ . \qquad \ldots(3.2.46.)$$

Putting the values from (3.2.35)-(3.2.38) in (3.2.34), we get.

$$[k\lambda(1-x)+\mu(1-z^b)]Q(x,z) = \mu\sum_{n=0}^{a-1}\sum_{r=1}^{k} Q_{n,r}(x-x^r)-k\lambda Q_{N,k}z^{N+1}$$

$$+\mu\sum_{r=1}^{k}\sum_{n=0}^{b-1} Q_{n,r}x^r(1-z^{n-b})$$

$$+k\lambda x Q_k(z)(z-x^k). \qquad \ldots(3.2.47.)$$

substituting $x = [1+(\frac{\mu}{k\lambda})-\frac{\mu}{k\lambda z^b}]$ in (3.2.39), we get

$$Q_k(z)=\frac{\mu}{k\lambda}\;\frac{\left[\begin{array}{l} -\sum\limits_{n=0}^{a-1}\sum\limits_{r=1}^{k} Q_{n,r}\left[1-(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b}\right]^{r-1} \\ +\sum\limits_{r=1}^{k}\sum\limits_{n=0}^{b-1} Q_{n,r}\left[1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b}\right]^{r-1}(z^{n-b}-1) \\ +\frac{k\lambda}{\mu}(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b})^{-1}z^{N+1}Q_{N,k} \end{array}\right]}{\left[z-(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b})^k\right]} \qquad \ldots(3.2.48.)$$

The zeros of the denominator of (3.2.40) can be obtained from the solution of the equation

$$z-(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z^b})^k = 0\,. \qquad \ldots(3.2.49.)$$

Applying Rouche's theorem to (3.2.41), we find that if $\rho = \lambda/b\mu$ is less than one, there are (bk−1) roots inside, one on and one root outside $|z|=1$ the proof that the single root (z_0) outside the unit circle is real is relatively straightforword, since $f(1)<0$ where as $f(\omega)>0$. Returning to (3.2.40) we note the numerator has bk zeros. Since $Q_k(z)$ exists the zero of the numerator and denominator must cancel leaving

$$Q_k(z) = \frac{A(z_0^{N+1} - z^{N+1})}{(z-z_0)\ z_0^{N+1}}$$

$$= \sum_{n=0}^{N} \frac{-A}{z_0^{n+1}} z^n .$$

Where the constant A is easily evaluated as $A = -z_0 Q_{0,k}$ and therefore the p.g.f. for a busy server takes the simple form

$$Q_k(z) = \frac{Q_{0,k}(z_0^{N+1} - z^{N+1})}{z_0^N (z_0 - z)} . \qquad \text{...(3.2.50.)}$$

Thus

$$Q_{n.k} = \frac{Q_{0,k}}{z_0^n} \qquad (0 \le n \le N) \qquad \text{...(3.2.51.)}$$

From (3.2.25), solving recursively, we get

$$Q_{n,r} = \left(1 + \frac{\mu}{k\ \lambda} - \frac{\mu}{k\ \lambda z_0^b}\right)^{k-r} Q_{n,k} . \qquad \text{...(3.2.52.)}$$

$$(1 \le r \le k\ ;\ 1 \le n \le N-b)$$

Solving (3.2.26) recursively, we get

$$Q_{n,r} = \left(1+\frac{\mu}{k\,\lambda}\right)^{k-r} Q_{n,k} \quad \ldots(3.2.53.)$$

$$(N-b+1 \le n \le N \;;\; 1 \le r \le k)$$

For r=1, relation (3.2.53) reduces to

$$Q_{n-1,k} = Q_{n,k}\left(1+\frac{\mu}{k\,\lambda}\right)^{k} \quad \ldots(3.2.54.)$$

$$(N-b+1 \le n \le N)$$

we finally determine $Q_{0,r}$, $1 \le r \le k$ consider (3.2.23)

$$(k\,\lambda+\mu)Q_{0,r} = k\,\lambda Q_{0,r-1} + \mu \sum_{j=a}^{b} Q_{j,r} \qquad (2 \le r \le k)$$

$$= k\,\lambda Q_{0,r-1} + \mu \sum_{j=a}^{b} \left[\frac{Q_{0,k}\left(1+\frac{\mu}{k\,\lambda} - \frac{\mu}{k\,\lambda z_0^{b}}\right)^{-r}}{z_0^{n-1}} \right]$$

$$= k\,\lambda\, Q_{0,r-1} + \frac{\mu}{k\,\lambda} \frac{(z_0^{b+1} - z_0^{a})Q_{0,k}}{(z_0-1)(z_0^{a+b-1})} \left(1+\frac{\mu}{k\,\lambda} - \frac{\mu}{k\,\lambda z_0^{b}}\right)^{-r}.$$

or

$$\left(1+\frac{\mu}{k\,\lambda}\right)Q_{0,r} = Q_{0,r-1} + \frac{\mu}{k\,\lambda} \frac{(z_0^{b+1} - z_0^{a})Q_{0,k}}{(z_0-1)z_0^{a+b-1}} \left(1+\frac{\mu}{k\,\lambda} - \frac{\mu}{k\,\lambda z_0^{b}}\right)^{-r}.$$

$$\ldots(3.2.55.)$$

Solving the above relation recursively for r=r+1,r+2,....k, we get

$$Q_{0,r}=\left[(1+\frac{\mu}{k\lambda})^{k-r}(z_0^a-z_0^b)+(z_0^{b+1}-z_0^a)(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b})^{-r}\right].$$

$$\frac{Q_{0,k}}{z_0^{a-1}(z_0-1)}. \qquad \ldots(3.2.56.)$$

In order to determine the value of $Q_{0,k}$, we first consider the idle probabilities using the steady state equations and defining

$$P_r(z)=\sum_{n=0}^{a-1} P_{n,r}z^n \quad (\text{ since for } n\geq a \text{ , } P_{n,r}=0) \qquad \ldots(3.2.57.)$$

$$P(x,y)=\sum_{r=1}^{k} P_r(z)x^r \qquad \ldots(3.2.58.)$$

To obtain the value of $P_k(z)$, solve (3.2.21.) recursively for $r=k$, $k-1$, ...2 we get

$$k\lambda P_{n,k} = k\lambda P_{n,1}+\mu\sum_{r=2}^{k} Q_{n,r} . \qquad \ldots(3.2.59.)$$

with the help of (3.2.20) the (3.2.59) reduces

$$k\lambda P_{n,k} = \mu\sum_{r=1}^{k} Q_{n,r}+k\lambda P_{n-1,k} . \qquad \ldots(3.2.60.)$$

Solving (3.2.60) recursively for $n=a-1$, $a-2$,... 2,1, we get.

$$k\lambda P_{a-1,k}=\mu\sum_{n=1}^{a-1} Q_n + k\lambda P_{0,k} . \qquad \ldots(3.2.61.)$$

Now for (3.2.60), we get

$$k\lambda\sum_{n=1}^{a-1} P_{n,k}z^n=\mu\sum_{n=1}^{a-1} Q_n z^n+zk\lambda\sum_{n=0}^{a-2} P_{n,k}z^n$$

or

$$k\lambda\sum_{n=0}^{a-1} P_{n,k}z^n-k\lambda P_{0,k}=\mu\sum_{n=1}^{a-1} Q_n z^n+zk\lambda\sum_{n=0}^{a-1} P_{n,k}z^n-zk\lambda P_{a-1,k}z^{a-1}$$

or

$$k\lambda(1-z)P_k(z) = \mu\sum_{n=1}^{a-1} Q_n(z^n-z^a)+(1-z^a)k\lambda P_{0,k} \quad . \quad \ldots(3.2.62.)$$

Now to obtain the value of $k\lambda P_{0,k}$ we solve (3.2.21) recursively for $r=k,k-1,k-2$, ... 2 after putting $n=0$

$$k\lambda P_{0,k} = \mu Q_0 \quad . \qquad \ldots(3.2.63.)$$

Now (3.2.62), with the help of (3.2.63) gives

$$P_k(z) = \mu\sum_{n=0}^{a-1} \frac{(z^n-z^a)Q_n}{k\lambda(1-z)} \quad . \qquad \ldots(3.2.64.)$$

It is possible to show, using the recursion redations to form p.gf's that

$$P_k(1) = P_{k-1}(1) = \ldots\ldots.=P_1(1) = 1/k \ .$$

where $P_r(z) = Q_r(z) + P_r(z)$.

Therefore we can say upon the substitution (3.2.64) and the use of L Hospital's rule that

$$\frac{1}{k} = Q_k(1) + P_k(1). \qquad \ldots(3.2.65.)$$

Putting z=1 in (3.2.50), we get

$$Q_k(1) = \frac{Q_{0,k}(z_0^{N+1}-1)}{z_0^N(z_0-1)} . \qquad \ldots(3.2.66.)$$

Now put z=1 in (3.2.64) and use L Hospital's rule, we get

$$P_k(1) = \frac{\mu}{k\lambda}\sum_{r=1}^{k} aQ_{0,r} + \frac{\mu}{k\lambda}\sum_{n=1}^{a-1}\sum_{r=1}^{k}(a-n)Q_{n,r} . \qquad \ldots(3.2.67.)$$

The value of the first term of (3.2.67) ofter taking the appropriate summation is

$$\frac{\mu}{k\lambda}a\sum_{r=1}^{k} Q_{0,r} = a\left[\frac{(z_0^a-z_0^b)}{z_0^{a-1}(z_0-1)}\{1-(1+\frac{\mu}{k\lambda})^k\} + \frac{(z_0^{b+1}-z_0^a)z_0^b}{z_0^a(z_0^b-1)}\right]Q_{0,k} . \qquad \ldots(3.2.68.)$$

The value of the second term of (3.2.67) is

$$\frac{\mu}{k\lambda}\sum_{n=1}^{a-1}\sum_{r=1}^{k}(a-n)Q_{n,r} = \frac{z_0^b\,Q_{0,k}}{(1-z_0^b)}\left[1-a+\frac{z_0^{a-1}-1}{z_0^{a-1}(z_0-1)}\right] . \qquad \ldots(3.2.69.)$$

Now substituting the values from (3.2.66)-(3.2.69) in (3.2.65) we get

$$\frac{1}{k}=\frac{Q_{0,k}(z_0^{N+1}-1)}{z^N\ (z_0-1)}+\frac{\mu}{k\ \lambda}\sum_{r=1}^{k} aQ_{0,r}+\frac{\mu}{k\ \lambda}\sum_{n=1}^{a-1}\sum_{r=1}^{k}(a-n)Q_{n,r} \quad \ldots(3.2.70.)$$

Solving (3.2.70), we get

$$Q_{0,k}=\frac{1}{k}\left[\left[\frac{(z_0^{N+1}-1)}{z_0^N(z_0-1)}-\frac{z_0^b}{z_0^b-1}\left[\frac{z_0^a-1}{z_0^{a-1}(z_0-1)}-a\right]+\right.\right.$$
$$\left.a\left[\frac{z_0^a-z_0^b}{z_0^{a-1}(z_0-1)}(1-(1+\frac{\mu}{k\ \lambda})^k)+\frac{z_0^b(z_0^{b+1}-z_0^a)}{z_0^a\ (z_0^b-1)}\right]\right]^{-1}. \quad \ldots(3.2.71.)$$

When specialized one gets the systems $M/M^{a,b}/1/N$ and $E_k/M^{a,b}/1/N$. (3.2.71) duplicates results obtained by Borthakur [22] and reported in Chaudhry and Templeton[ ] respectively. Retering to the steady state equations, we determine the remaining probabilities recursively.

To obtain the value of $P_{a-1,k}$ consider (3.2.52)

$$Q_{n,r}=(1+\frac{\mu}{k\ \lambda}-\frac{\mu}{k\ \lambda z_0^b})^{k-r}Q_{n,k}\ .$$

$$(1\le r\le k\ ;\ 1\le n\le N-b)$$

put $r=1$

$$Q_{n,1}=\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)^{k-1}\frac{Q_{0,k}}{z_0^n}\,. \qquad \ldots(3.2.72.)$$

now from (3.2.22), We obtain with the help of (3.2.72)

$$P_{a-1,k}=-\frac{\mu}{k\lambda}\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)^{k-1}Q_{0,k}\sum_{n=a}^{b}\frac{1}{z_0^n}$$

$$+\left(1+\frac{\mu}{k\lambda}\right)Q_{0,1}\,. \qquad \ldots(3.2.73.)$$

$$\sum_{n=a}^{b}\frac{1}{z_0^n}=\frac{z_0^{1-a}-z_0^{-b}}{(z_0-1)}\,. \qquad \ldots(3.2.74.)$$

with the help of (3.2.74),(3.2.73) reduces

$$P_{a-1,k}=-\frac{\mu}{k\lambda}\left[\frac{z_0(z_0^{1-a}-z_0^{-b})Q_{0,k}}{\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)(z_0-1)}\right]$$

$$+\left(1+\frac{\mu}{k\lambda}\right)Q_{0,1}\,. \qquad \ldots(3.2.75.)$$

To obtain the value of $P_{n,r}$ solve (3.2.21) recursively for $r=r+1$, $r+2$, ...,k , we get

$$k\lambda P_{n,k}=\mu\sum_{j=r+1}^{k}Q_{n,j}+k\lambda P_{n,r}\,. \qquad \ldots(3.2.76.)$$

or

$$P_{n,r} = P_{n,k} - \frac{\mu}{k\lambda} \sum_{j=r+1}^{k} Q_{n,j}$$

$$= P_{n,k} - \frac{\mu}{k\lambda} \sum_{j=0}^{k-r-1} \left(1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{k-j-r} \frac{Q_{0,k}}{z_0^n}$$

or

$$P_{n,r} = P_{n,k} + \frac{z_0^b \left[1 - z_0 \left(1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-r}\right]}{z_0^n \,(z_0^b - 1)} Q_{0,k} \quad \ldots\ldots(3.2.77.)$$

$$(1 \leq r \leq k-1 \;;\; 0 \leq n \leq a-1)$$

To obtain the value of $P_{n,k}$, put $r=1$ in (3.2.77), we get

$$P_{n,1} = P_{n,k} + \frac{z_0^b \left[1 - z_0 \left(1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-1}\right]}{z_0^n (z_0^b - 1)} Q_{0,k} \;. \quad \ldots(3.2.78.)$$

Putting the value of $P_{n,1}$ from (3.2.78) in (3.2.20), we get

$$P_{n,k} = \frac{-Q_{0,k} z_0^b \left[1 - z_0 \left(1+\frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-1}\right]}{z_0^n (z_0^b - 1)} +$$

$$\frac{\mu}{k\lambda} Q_{n,1} + P_{n-1,k} \;. \quad \ldots(3.2.79.)$$

Solving (3.2.79) recursively for $n=a-1,\ a-2,\ \ldots\ldots\ n+1$, we get

$$P_{n,k}=P_{a-1,k}+\frac{Q_{0,k}z_0^b\left[1-z_0\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)^{-1}\right]}{z_0^b-1}\left[\frac{1}{z_0^{a-1}}+\frac{1}{z_0^{a-2}}+\ldots+\frac{1}{z_0^{n+1}}\right]$$

$$-\frac{\mu}{k\lambda}\sum_{n=a-1}^{n+1}Q_{n,1}\ . \qquad \ldots(3.2.80.)$$

The value of the second term of (3.2.80) is obtained as

$$\frac{Q_{0,k}z_0^b\left[1-z_0\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)^{-1}\right]\{z_0-(z_0)^{a-n}\}}{z_0^{n+a}(z_0^b-1)(1-z_0)}\ ,$$

$$\ldots(3.2.81.)$$

The value of the third term of r.h.s. of (3.2.80) is obtained as

$$\frac{\mu}{k\lambda}\sum_{n=a-1}^{n+1}Q_{n,1}=\frac{\mu}{k\lambda}\left[1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right]^{-1}Q_{0,k}z_0^{a-2}\ .$$

$$\left[\frac{1-(z_0)^{n-a+3}}{1-z_0}\right]. \qquad \ldots(3.2.82.)$$

Putting the values from (3.2.81) and (3.2.82) in (3.2.80), we get

$$P_{n,k}=P_{a-1,k}+\frac{z_0^b\left[1-z_0\left(1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right)^{-1}\right]\{z_0-(z_0)^{a-n}\}}{z_0^{n+a}(z_0^b-1)(1-z_0)}$$

$$-\frac{\mu}{k\lambda}\left[1+\frac{\mu}{k\lambda}-\frac{\mu}{k\lambda z_0^b}\right]^{-1}Q_{0,k}\left[\frac{z_0^{a-2}-z_0^{n+1}}{(1-z_0)}\right]\ldots(3.2.83.)$$

3.3 IMBEDDED MARKOV CHAIN RESULTS

Let us consider the relation ship of the random point probabilities to those of the arrival epoch

Define :

$\bar{Q}_n$ = probability of n in queue at arrival epoch and server busy we can see, using the standard procedure.

$$\bar{Q}_n = kQ_{n,k} \; ; \; n \geq 0 \; . \qquad \ldots(3.3.1.)$$

Similarly if one considers the idle server case

$$\bar{P}_n = kP_{n,k} \; ; \; 0 \leq n \leq a-1 \; . \qquad \ldots(3.3.2.)$$

we have already obtained the values of $Q_{n,k}$ and $P_{n,k}$

3.4 MOMENTS (QUEUE LENGTH)

Define :

L_q : (the expected number in the queue at random epoch)

$\bar{L}_q$: (the expected number in the queue at arrival epoch)

$$L_q = \sum_{n=0}^{N} \sum_{r=1}^{k} nQ_{n,r} + \sum_{n=0}^{a-1} \sum_{r=1}^{k} n\, P_{n,r} \, . \qquad \ldots(3.4.1.)$$

and

$$\bar{L}_q = \sum_{n=0}^{N} n\bar{Q}_n + \sum_{n=0}^{a-1} n\, \bar{P}_n$$

or

$$\bar{L}_q = k \left[\sum_{n=0}^{N} nQ_{n,k} + \sum_{n=0}^{a-1} n\, P_{n,k} \right] . \qquad \ldots(3.4.2.)$$

The values of two terms on r.h.s. of (3.4.1) & (3.4.2) can be obtained by usual mathod.

Sample listings of the characteristic roots of z_0 for the bulk serive queue $E_k/M^{a,b}/1/N$

			k = 8		
RHO b =	1	2	3	4	5
0.050000	1.250000	1.154149	1.105273	1.073481	1.050683
0.100000	1.250000	1.154469	1.105853	1.074308	1.051751
0.150000	1.250000	1.154806	1.106501	1.075299	1.053168
0.200000	1.250000	1.155165	1.107236	1.076549	1.055308
0.250000	1.250000	1.155549	1.108093	1.078259	1.059693
0.300000	1.250000	1.155962	1.109124	1.080955	1.115619
0.350000	1.250000	1.156409	1.110418	1.086818	1.053293
0.400000	1.250000	1.156897	1.112150	1.140712	1.103352
0.450000	1.250000	1.157434	1.114706	1.064911	1.040512
0.500000	1.250000	1.158032	1.119171	1.051628	1.051628

			k = 9		
RHO b =	1	2	3	4	5
0.050000	1.250000	1.160976	1.115155	1.084971	1.062932
0.100000	1.250000	1.161251	1.115661	1.085703	1.063898
0.150000	1.250000	1.161542	1.116226	1.086577	1.065165
0.200000	1.250000	1.161852	1.116866	1.087674	1.067039
0.250000	1.250000	1.162183	1.117612	1.089159	1.070685
0.300000	1.250000	1.162539	1.118506	1.091466	1.096646
0.350000	1.250000	1.162925	1.119627	1.096299	1.064228
0.400000	1.250000	1.163346	1.121123	1.130383	1.098813
0.450000	1.250000	1.163810	1.123324	1.076418	1.052587
0.500000	1.250000	1.164326	1.127150	1.088317	1.063683

CHAPTER 4

4.1 INTRODUCTION

Kambo and Chaudhry[105]considered a bulk service queueing system $E_k/M^{a,b}/1$ and discussed a distribution of busy period for the system. In this paper they considered a situation in which interarrival times of units have an Erlang distribution with a mean for each phase of $1/k\lambda$, (k being the number of phases) such that the over all mean inter arrival time is $1/\lambda$. The units entering the system join a single queue of unlimited size. The bulk service govern by a general rule. Service begins only when a units are presents. A queue length is $\geq$ a but $\leq$ b the entire queue is teken up for service; if there are more than b units in queue then the server accepts the first b units. Such a service rule, was first discussed by Nutes [146] and letter by Holman [90] and others.

The study of busy period distribution have been impartant from the server's point of view . Various authors studied busy period distribution for different queueing models. In particular Bar Thakur studied busy period distribution for the system $M^x/G^{a,b}/1$.

In the present chapter we are discussing a situation similar, that discussed by Kambo and Chaudhry[105]. In which the units entering the system join a single queue of limited size say M, and the interarrival times of the units have a general distribution with density function A(x). Such situations have been discussed in past in connection with double ended queueing models discussed by Kashyap[106] and Gaur[51].

In 4.2 of this chapter we discuss the busy period distribution of the system and the Laplace transform of the general probability $P_{n,r,1}(x,t)$ is obtained.

4.2 BUSY PERIOD EQUATIONS

Define :

$P_{n,r,1}(x,t)$ = Probability that at time $t(\geq 0)$ the queue size is $n(\geq 0)$ the arriaval is in phase $r(1\leq r<k)$, and the server is busy;

$P_{n,r,0}(x,t)$ = Probability that at time $t(\geq 0)$ the queue size is $n(0\leq n\leq a-1)$ the arriavle is in phase $r(1\leq r<k)$, and the server is idle.

The fundamental differential equations descriving the busy period are obetained as follows;

$$P_{0,k,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{0,k,1}(x,t)+\mu\Delta P_{a,1,1}(x,t)$$
$$+\mu\Delta\sum_{j=a}^{b}P_{j,k,1}(x,t), \qquad \ldots(4.2.1.)$$

$$P_{nr,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{0,r,1}(x,t)+\mu\Delta tP_{0,r+1,1}(x,t)$$
$$+\mu\Delta t\sum_{j=a}^{b}p_{j,r,1}(x,t), \qquad \ldots(4.2.2.)$$

$$P_{n,k,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{n,k,1}(x,t)+\mu\Delta tP_{n+b,k,1}(x,t),$$
$$(1\leq n\leq n-b) \qquad \ldots(4.2.3.)$$

$$P_{n,r,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]+\mu\Delta t\ P_{n,r+1,1}(x,t)$$
$$+\ \mu\Delta t P_{n+b,r,1}(x,t), \qquad \ldots(4.2.4.)$$

$$(1\leq n\leq n-b\ ;\ 1\leq r<k)$$

$$P_{n,k,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{n,k,1}(x,t)$$

$$+\mu\Delta t P_{n+1,1,1}(x,t), \qquad \ldots(4.2.5.)$$

$$(N-b+1\leq n<n)$$

$$P_{n,r,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{n,r,1}(x,t)$$

$$+\mu\Delta t P_{n,r+1,1}(x,t)+\mu\Delta t P_{n+1,1,1}(x,t),$$

$$\ldots(4.2.6.)$$

$$(1\leq r<k\ ;\ N-b+1\leq n<N)$$

$$P_{N,k,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{N,k,1}(x,t),$$

$$\ldots(4.2.7.)$$

$$P_{N,r,1}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t][1-\mu\Delta t]P_{N,r,1}(x,t),$$

$$+\mu\Delta t P_{N,r+1,1}(x,t) \qquad \ldots(4.2.8.)$$

$$(1\leq r<k)$$

$$P_{0,1,0}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t]P_{0,1,0}(x,t), \qquad \ldots(4.2.9.)$$

$$P_{n,k,0}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t]P_{n,k,0}(x,t)$$

$$+\mu\Delta(1-\eta(x)\Delta t)P_{n+1,1,0}(x,t), \quad \ldots(4.2.10.)$$

$$P_{n,r,0}(x+\Delta x,t+\Delta t)=[1-\eta(x)\Delta t]P_{n,r,0}(x,t)$$

$$+\mu\Delta t(1-\eta(x)\Delta t)P_{n,r+1,0}(x,t). \quad \ldots(4.2.11.)$$

$$(0<n\leq a-1 \;;\; 0\leq r<k)$$

Transposing and letting $\Delta\to 0$, (4.2.1)-(4.2.11) transform to the form :

$$\frac{\partial}{\partial t} P_{0,k,1}(x,t)+\frac{\partial}{\partial x} P_{0,k,1}(x,t)+(\eta(x)+\mu)P_{0,k,1}(x,t)$$

$$=\mu P_{a,1,1}(x,t)+\mu\sum_{j=a}^{b} P_{j,k,1}(x,t), \quad \ldots(4.2.12.)$$

$$\frac{\partial}{\partial t}P_{0,r,1}(x,t)+\frac{\partial}{\partial x}P_{0,r,1}(x,t)+(\eta(x)+\mu)P_{0,r,1}(x,t)$$

$$=\mu P_{0,r+1,1}(x,t)+\mu\sum_{j=a}^{b} P_{0,r,1}(x,t), \quad \ldots(4.2.13.)$$

$$\frac{\partial}{\partial t}P_{n,k,1}(x,t)+\frac{\partial}{\partial x}P_{n,k,1}(x,t)+(\eta(x)+\mu)P_{n,k,1}(x,t)$$

$$+\mu P_{n+b,k,1}(x,t), \quad \ldots(4.2.14.)$$

$$(1\leq n\leq N-b)$$

$$\frac{\partial}{\partial t}P_{n,r,1}(x,t)+\frac{\partial}{\partial x}P_{n,r,1}(x,t)+(\eta(x)+\mu)P_{n,r,1}(x,t)$$

$$=\mu P_{n,r+1,1}(x,t)+\mu P_{n+b,r,1}(x,t), \quad \ldots(4.2.15.)$$

$$(1\le n\le N-b \;;\; 1\le r<k)$$

$$\frac{\partial}{\partial t}P_{n,k,1}(x,t)+\frac{\partial}{\partial x}P_{n,k,1}(x,t)+(\eta(x)+\mu)P_{n,k,1}(x,t)$$

$$=\mu P_{n+1,1,1}(x,t), \quad \ldots(4.2.16.)$$

$$(N-b+1 \le \eta<N)$$

$$\frac{\partial}{\partial t}P_{n,r,1}(x,t)+\frac{\partial}{\partial x}P_{n,r,1}(x,t)+(\eta(x)+\mu)P_{n,r,1}(x,t)$$

$$=\mu P_{n,r+1,1}(x,t)+\mu P_{n+1,1,1}.(x,t), \ldots(4.2.17.)$$

$$(1\le r<k \;;\; N-b+1 \le n<N)$$

$$\frac{\partial}{\partial t}P_{N,k,1}(x,t)+\frac{\partial}{\partial x}P_{N,k,1}(x,t)+(\eta(x)+\mu)P_{N,k,1}(x,t) = 0,$$

$$\ldots(4.2.18.)$$

$$\frac{\partial}{\partial t}P_{N,r,1}(x,t)+\frac{\partial}{\partial x}P_{N,r,1}(x,t)+(\eta(x)+\mu)P_{N,r,1}(x,t)$$

$$=\mu P_{N,r+1,1}(x,t), \quad \ldots(4.2.19.)$$

$$(1\le r<k)$$

$$\frac{\partial}{\partial t}P_{0,1,0}(x,t)+\frac{\partial}{\partial x}P_{0,1,0}(x,t)+\eta(x)P_{0,1,0}(x,t)=0, \quad \text{...(4.2.20.)}$$

$$\frac{\partial}{\partial t}P_{n,k,0}(x,t)+\frac{\partial}{\partial x}P_{n,k,0}(x,t)+\eta(x)P_{n,k,0}(x,t)$$

$$=\mu P_{n+1,1,0}(x,t), \quad \text{...(4.2.21.)}$$

$$\frac{\partial}{\partial t}P_{n,r,0}(x,t)+\frac{\partial}{\partial x}P_{n,r,0}(x,t)+\eta(x)P_{n,r,0}(x,t)$$

$$=\mu P_{n,r+1,0}(x,t). \quad \text{...(4.2.22.)}$$

$$(0<\eta\leq a-1 \; ; \; 0\leq r<k)$$

We define the following generating functions

$$P_{n,1}(x,y,t)=\sum_{r=1}^{k} P_{n,r,1}(x,t)y^{r} , \quad \text{...(4.2.23.)}$$

$$P_{1}(x,y,z,t)=\sum_{n=0}^{N} P_{n,1}(x,y,t)z^{n} , \quad \text{...(4.2.24.)}$$

$$P_{n,r,1}(x,0)=\delta_{in}\,\delta_{mr}\,\delta(x). \quad \text{...(4.2.25.)}$$

$$P_{1}(x,y,z,0)=y^{i}\,z^{m}\,\delta(x) . \quad \text{...(4.2.26.)}$$

Following the usual technique of generating-function analysis from equations (12)-(19),we get.

$$P_1(x,y,z,t) = \sum_{n=0}^{N}\sum_{r=1}^{k} P_{n,r,1}(x,t)y^r z^n . \qquad \ldots(4.2.27.)$$

$$\frac{\partial}{\partial t}P_1(x,y,z,t) = \frac{\partial}{\partial t}\left[\sum_{n=0}^{N}\sum_{r=1}^{k} P_{n,r,1}(x,t)z^n\right]$$

$$= \frac{\partial}{\partial t}\sum_{r=1}^{k} y^r\left[P_{0,r,1}(x,t)+\sum_{n=1}^{N-b} z^n P_{n,r,1}(x,t) + \sum_{n=N-b+1}^{N} P_{n,r,1}(x,t)z^n + z^N P_{N,r,1}(x.t)\right].$$

$$\ldots(4.2.28.)$$

Now taking the appropriate summations of each term of (4.2.28.) we get,

$$\sum_{r=1}^{k} y^r \frac{\partial}{\partial t}P_{0,r,1}(x,t) = \mu\sum_{r=1}^{k-1} y^r P_{0,r+1,1}(x,t)$$

$$+\mu\sum_{r=1}^{k} y^r \sum_{j=a}^{b} P_{j,r,1}(x,t)$$

$$-\sum_{r=1}^{k} y^r \frac{\partial}{\partial x}P_{0,r,1}(x,t)+\mu y^k P_{a,1,1}(x,t)$$

$$-(\eta(x)+\mu)\sum_{r=1}^{k} y^r P_{0,r,1}(x,t),$$

$$\ldots(4.2.29.)$$

$$\sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n \frac{\partial}{\partial t} P_{n,r,1}(x,t) = \mu \sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n P_{n+b,r,1}(x,t)$$

$$+\mu \sum_{r=1}^{k-1} y^r \sum_{n=1}^{N-b} z^n P_{n,r+1,1}(x,t)$$

$$-\sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n \frac{\partial}{\partial x} P_{n,r,1}(x,t)$$

$$-(\eta(x)+\mu) \sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n P_{n,r,1}(x,t),$$

$$\ldots(4.2.30.)$$

$$\sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N} z^n \frac{\partial}{\partial t} P_{n,r,1}(x,t) = \mu \sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N-1} z^n P_{n,r+1,1}(x,t)$$

$$+\mu \sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N-1} z^n P_{n+1,1,1}(x,t)$$

$$-\sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N-1} z^n \frac{\partial}{\partial x} P_{n,r,1}(x,t)$$

$$-(\eta(x)+\mu) \sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N-1} z^n P_{n,k,1}(x,t),$$

$$\ldots(4.2.31)$$

$$\sum_{r=1}^{k} y^r \frac{\partial}{\partial t} P_{N,r,1}(x,t) z^N = \mu \sum_{r=1}^{k-1} y^r z^N P_{N,r+1,1}(x,t)$$

$$-\sum_{r=1}^{k} y^r z^N \frac{\partial}{\partial x} P_{N,r,1}(x,t)$$

$$-(\eta(x)+\mu) \sum_{r=1}^{k} y^r z^N P_{N,r,1}(x,t).$$

...(4.2.32.)

Putting the values from equtions (4.2.29)-(4.2.32) in (4.2.28), we get

$$\frac{\partial}{\partial x} P_1(x,y,z,t) + \frac{\partial}{\partial x} P_1(x,y,z,t) + [\eta(x)+\mu] P_1(x,y,z,t)$$

$$= \mu \sum_{n=0}^{N} z^n \sum_{r=1}^{k-1} y^r P_{n,r+1,1}(x,t)$$

$$+\mu \sum_{r=1}^{k} y^r \sum_{n=a}^{b} P_{n,r,1}(x,t)$$

$$+\mu \sum_{r=1}^{k} \sum_{n=N-b+1}^{N-1} z^n P_{n+1,1,1}(x,t)$$

$$+\mu y^k P_{a,1,1}(x,t) + \mu \sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n P_{n+b,r,1}(x,t).$$

...(4.2.33.)

obtaining the values of certain terms in (4.2.33) by usual methods, we have

$$\mu\sum_{r=1}^{k} y^r \sum_{n=1}^{N-b} z^n P_{n+b,r,1}(x,t) = \mu z^{-b} P_1(x,y,z,t) -$$

$$\mu z^{-b} \sum_{r=1}^{k} y^r \sum_{n=0}^{b} P_{n,r,1}(x,t) z^n,$$

$$\ldots\text{(i)}$$

$$\mu\sum_{n=0}^{N} z^n \sum_{r=1}^{k-1} y^r P_{n,r+1,1}(x,t) = \mu y P_1(x,y,z,t), \qquad \ldots\text{(ii)}$$

$$\mu\sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N-1} z^n P_{n+1,1,1}(x,t) = \mu z \sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N} z^n P_{n,1,1}(x,t),$$

$$\ldots\text{(iii)}$$

Putting the values of above three terms in (4.2.33) we get

$$\frac{\partial}{\partial t} P_1(x,y,z,t) + \frac{\partial}{\partial x} P_1(x,y,z,t) + \{\eta(x) + \mu - \mu z^{-b} - \mu y\} P_1(x,y,z,t)$$

$$= \mu \sum_{r=1}^{k} y^r \sum_{n=a}^{b} P_{n,r,1}(x,t) + \mu z \sum_{r=1}^{k} y^r \sum_{n=N-b+1}^{N} z^n P_{n,1,1}(x,t)$$

$$+ \mu y^k P_{a,1,1}(x,t) - \mu z^{-b} \sum_{r=1}^{k} y^r \sum_{n=0}^{b} P_{n,r,1}(x,t) z^n.$$

$$\ldots(4.2.34.)$$

Taking Laplace transform, we have

$$\frac{\partial}{\partial x}\overline{P}_1(x,y,z,s)+\{s+\mu+\eta(x)-\mu z^{-b}-\mu y\}\overline{P}_1(x,y,z,s)$$

$$=\mu\sum_{r=1}^{k} y^r\sum_{n=a}^{b}\overline{P}_{n,r,1}(x,s)+\mu z\sum_{r=1}^{k} y^r\sum_{n=N-b+1}^{N} z^n P_{n,1,1}(x,s)$$

$$+\mu y^k\overline{P}_{a,1,1}(x,s)-\mu z^{-b}\sum_{r=1}^{k} y^r\sum_{n=0}^{b}\overline{P}_{n,r,1}(x,s)z^n+y^i z^m\delta(x).$$

...(4.2.35.)

Similarly (4.2.12)-(4.2.22) yield

$$\frac{\partial}{\partial x}\overline{P}_{0,k,1}(x,s)+\{s+\mu+\eta(x)\}\overline{P}_{0,k,1}(x,s) = \mu\overline{P}_{a,1,1}(x,s)+\mu\sum_{j=a}^{b}\overline{P}_{j,k,1}(x,s),$$

...(4.2.36.)

$$\frac{\partial}{\partial x}\overline{P}_{0,r,1}(x,s)+\{s+\mu+\eta(x)\}\overline{P}_{0,r,1}(x,s)=\mu\overline{P}_{0,r+1,1}(x,s)+\mu\sum_{j=a}^{b}\overline{P}_{j,r,1}(x,s),$$

...(4.2.37.)

$$\frac{\partial}{\partial x}\overline{P}_{n,k,1}(x,s)+\{s+\mu+\eta(x)\}\overline{P}_{n,k,1}(x,s) = \mu\overline{P}_{n+b,k,1}(x,s),$$

$(1\leq n\leq N-b)$ ...(4.2.38.)

$$\frac{\partial}{\partial x}\bar{P}_{n,r,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{n,r,1}(x,s) = \mu\bar{P}_{n,r+1,1}(x,s)+\mu\bar{P}_{n+b,r,1}(x,s)$$

$$+\delta_{in}\delta_{mr}\delta(x),$$

$$(1\le n\le N-b \;;\; 1\le r\le k) \qquad \ldots(4.2.39.)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,k,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{n,k,1}(x,s) = \mu\bar{P}_{n+1,1,1}(x,s)$$

$$(N-b+1\le n<N) \qquad \ldots(4.2.40.)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,r,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{n,r,1}(x,s) = \mu\bar{P}_{n,r+1,1}(x,s)$$

$$+\mu\bar{P}_{n+1,1,1}(x,s),$$

$$(1\le r\le k \;;\; N-b+1\le n<N) \qquad \ldots(4.2.41.)$$

$$\frac{\partial}{\partial x}\bar{P}_{N,k,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N,k,1}(x,s) = 0, \qquad \ldots(4.2.42.)$$

$$\frac{\partial}{\partial x}\bar{P}_{N,r,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N,r,1}(x,s) = \mu\bar{P}_{N,r+1,1}(x,s),$$

$$(1\le r<k) \qquad \ldots(4.2.43.)$$

$$\frac{\partial}{\partial x}\bar{P}_{0,1,0}(x,s)+\{s+\eta(x)\}\bar{P}_{0,1,0}(x,s) = 0, \qquad \ldots(4.2.44.)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,k,0}(x,s)+\{s+\eta(x)\}\bar{P}_{n,k,0}(x,s) = \mu\bar{P}_{n+1,1,0}(x,s),$$

$$\ldots(4.2.45.)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,r,0}(x,s)+\{s+\eta(x)\}\bar{P}_{n,r,0}(x,s) = \mu\bar{P}_{n,r+1,1}(x,s).$$

$$(0<n\leq a-1 \; ; \; 0\leq r<k) \qquad \ldots(4.2.46.)$$

The solution of (4.2.42) is

$$\bar{P}_{N,k,1}(x,s) = \bar{P}_{N,k,1}(o,s)\, e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}. \qquad \ldots(4.2.47.)$$

Taking r = k-1 in (4.2.43) and using (4.2.47), we have

$$\frac{\partial}{\partial x}\bar{P}_{N,k-1,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N,k-1,1}(x,s)$$

$$= \mu\bar{P}_{N,k,1}(o,s)\, e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}. \qquad \ldots(4.2.48.)$$

Whence

$$\bar{P}_{N,k-1,1}(x,s) = e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}\, [\mu x\bar{P}_{n,k,1}(o,s)+\bar{P}_{N,k-1,1}(o,s)].$$

$$\ldots(4.2.49.)$$

Similarly taking r=k-2 in (4.2.43) and using (4.2.49) and solving the resulting differential equation, we have.

$$\bar{P}_{N,k-2,1}(x,s) = e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\frac{(\mu x)^2}{2!}\bar{P}_{N,k,1}(o,s) + \mu x\bar{P}_{n,k-1,1}(o,s) + \bar{P}_{n,k-2,1}(o,s) \right].$$

...(4.2.50.)

Proceeding similarly, when r=1, we have

$$\bar{P}_{N,1,1}(x,s) = e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\bar{P}_{N,1,1}(o,s)+\mu x\bar{P}_{N,2,1}(o,s) + \frac{(\mu x)^2}{2!}\bar{P}_{N,3,1}(o,s)+ \dots \frac{(\mu x)k-3}{(k-3)!}\bar{P}_{N,k-2,1}(o,s)+ \frac{(\mu x)^{k-2}}{(k-2)!}\bar{P}_{N,k-1,1}(o,s) + \frac{(\mu x)^{k-1}}{(k-1)!}\bar{P}_{N,k,1}(o,s) \right].$$

...(4.2.51.)

Taking n=N-1 in (4.2.40) and using (4.2.51) and solving the resulting differential equation, we get

$$\frac{\partial}{\partial x}\bar{P}_{N-1,k,1}(x,s)+(s+\mu+\eta(x))\bar{P}_{N-1,k,1}(x,s) = \mu\bar{P}_{N,1,1}(x,s).$$

...(4.2.52.)

$$\bar{P}_{N-1,k,1}(x,s) = e^{-(\mu+s)x} e^{-\int_0^x \eta(\mu)du} \left[\sum_{r=1}^{k} \frac{(\mu x)^r}{r!} \bar{P}_{N,r,1}(o,s) + \bar{P}_{N-1,k,1}(o,s) \right].$$

...(4.2.53.)

Now taking $n=N-1$ and $r=k-1$ in (4.2.41)

$$\frac{\partial}{\partial x}\bar{P}_{N-1,k-1,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N-1,k-1,1}(x,s)=\mu\bar{P}_{N-1,k,1}(x,s) +\mu\bar{P}_{N,1,1}(x,s).$$

...(4.2.54.)

Solving (4.2.54) and using (4.2.51) and (4.2.53) we get

$$\bar{P}_{N-1,k-1,1}(x,s) = e^{-(\mu+s)x} e^{-\int_0^x \eta(\mu)du} \left[\sum_{r=1}^{k} \frac{(\mu x)^{r+1}}{(r+1)!} \bar{P}_{N,r,1}(o,s) + \sum_{r=1}^{k} \frac{(\mu x)^r}{r!} \bar{P}_{N,r,1}(o,s) + \mu x\bar{P}_{N-1,k,1}(o,s) + \bar{P}_{N-1,k-1,1}(o,s) \right].$$

...(4.2.55.)

Similarly taking n=N-1 and r=k-2 in (4.2.41) and solving the resulting differential equation using (4.2.55), we get

$$\bar{P}_{N-1,k-2,1}(x,s) = e^{-(\mu+s)}\, e^{-\int_0^x \eta(\mu)du} \left[\sum_{r=1}^{k} \frac{(\mu x)^{r+2}}{(r+2)!} \bar{P}_{N,r,1}(o,s) + \sum_{r=1}^{k} \frac{(\mu x)^{r+1}}{(r+1)!} \bar{P}_{N,r,1}(o,s) + \sum_{r=1}^{k} \frac{(\mu x)^{r}}{r!} \bar{P}_{N,r,1}(o,s) + \frac{(\mu x)^2}{2!} \bar{P}_{N-1,k,1}(o,s) + \mu x \bar{P}_{N-1,k-1,1}(o,s) + \bar{P}_{N-1,k-2,1}(o,s) \right].$$

...(4.2.56.)

Proceeding similarly when n=N-1, r=k-m, we get

$$\bar{P}_{N-1,k-m,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\sum_{j=0}^{m} \sum_{r=1}^{k} \frac{(\mu x)^{r+j}}{(r+j)!} \bar{P}_{N,r,1}(o,s) + \sum_{j=0}^{m} \frac{(\mu x)^{j}}{(j)!} \bar{P}_{N-1,k-m+j,1}(o,s) \right].$$

...(4.2.57.)

Putting m=k-1 in (4.2.57), we get

$$\bar{P}_{N-1,1,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\sum_{r=1}^{k} \sum_{r_1=1}^{k} \frac{(\mu x)^{r+r_1-1}}{(r+r_1-1)!} \bar{P}_{N,r,1}(o,s) + \sum_{r=1}^{k} \frac{(\mu x)^{r-1}}{(r-1)!} \bar{P}_{N-1,r,1}(o,s) \right].$$

...(4.2.58.)

In (4.2.40) put n=N-2

$$\frac{\partial}{\partial x}\bar{P}_{N,-2,k,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N-2,k,1}(x,s)=\mu\bar{P}_{N-1,1,1}(x,s).$$

...(4.2.59.)

Solving (4.2.59) and using (4.2.58), we get

$$\bar{P}_{N-2,k,1}(x,s)=e^{-(\mu+s)x}e^{-\int_0^x \eta(\mu)du}\left[\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1}}{(r+r_1)!}\bar{P}_{N,r,1}(o,s)+\sum_{r=1}^{k}\frac{(\mu x)^r}{r!}\bar{P}_{N-1,r,1}(o,s)+\bar{P}_{N-2,k,1}(o,s)\right].$$

...(4.2.60.)

Now taking n=N-2 and r=k-1 in (4.2.41)

$$\frac{\partial}{\partial x}\bar{P}_{N,-2,k-1,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N-2,k-1,1}(x,s)=\mu\bar{P}_{N-2,k,1}(x,s)+\mu\bar{P}_{N-1,1,1}(x,s).$$

...(4.2.61.)

Solving (4.2.61) and using (4.2.58) & (4.2.60)

$$\bar{P}_{N-2,k-1,1}(x,s)=e^{-(\mu+s)}\, e^{-\int_0^x \eta(u)du}\Bigg[\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1+1}}{(r+r_1)!}\bar{P}_{N,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\frac{(\mu x)^{r+1}}{(r+1)!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\mu x\bar{P}_{N-2,k,1}(o,s)$$

$$+\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1}}{(r+r_1)!}\bar{P}_{N,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\frac{(\mu x)^{r}}{r!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\bar{P}_{N-2,k-1,1}(o,s)\Bigg].$$

Proceeding similarly, we get

$$\bar{P}_{N-2,k-2,1}(x,s)=e^{-(\mu+s)x}\,e^{-\int_0^x \eta(u)du}\Bigg[\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1}}{(r+r_1)!}\bar{P}_{N,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1+1}}{(r+r_1+1)!}\bar{P}_{N,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1+2}}{(r+r_1+2)!}\bar{P}_{N,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\frac{(\mu x)^{r}}{r!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\frac{(\mu x)^{r+1}}{(r+1)!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\sum_{r=1}^{k}\frac{(\mu x)^{r+2}}{(r+2)!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\frac{(\mu x)^2}{2!}\bar{P}_{N-2,k,1}(o,s)$$

$$+\mu x\bar{P}_{N-2,k-1,1}(o,s)$$

$$+\bar{P}_{N-2,k-2,1}(o,s)\Bigg].$$

...(4.2.62.)

In (4.2.41) taking $n=N-2$ and $r=k-m$, we get

$$\bar{P}_{N-2,k-m,1}(x,s)=e^{-(\mu+s)x-\int_0^x \eta(\mu)du}\left[\sum_{j=0}^{m}\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1+j}}{(r+r_1+j)!}\bar{P}_{N,r,1}(o,s)+\sum_{j=1}^{m}\sum_{r=1}^{k}\frac{(\mu x)^{r+j}}{(r+j)!}\bar{P}_{N-1,r,1}(o,s)+\sum_{j=0}^{m}\frac{(\mu x)^{j}}{(j)!}\bar{P}_{N-2,k-m+i,1}(o,s)\right].$$

...(4.2.63.)

Putting m=k-1 in (4.2.63), we get

$$\bar{P}_{N-2,1,1}(x,s)=e^{-(\mu+s)-\int_0^x \eta(\mu)du}\left[\sum_{r=1}^{k}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\frac{(\mu x)^{r+r_1+r_2-1}}{(r+r_1+r_2-1)!}\bar{P}_{N,r,1}(o,s)+\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1-1}}{(r+r_1-1)!}\bar{P}_{N-1,r,1}(o,s)+\sum_{r=1}^{k}\frac{(\mu x)^{r-1}}{(r-1)!}\bar{P}_{N-2,r,1}(o,s)\right]$$

...(4.2.64.)

Proceeding similarly, we get

$\bar{P}_{N-3,k-m,1}(x,s)$

$$=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}\left[\sum_{j=0}^{m}\sum_{r=1}^{k}\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\frac{(\mu x)^{r+r_1+r_2+j}}{(r+r_1+r_2+j)!}\bar{P}_{N,r,1}(o,s)\right.$$

$$+\sum_{j=0}^{m}\sum_{r=1}^{k}\sum_{r_1=1}^{k}\frac{(\mu x)^{r+r_1+j}}{(r+r_1+j)!}\bar{P}_{N-1,r,1}(o,s)$$

$$+\sum_{j=0}^{m}\sum_{r=1}^{k}\frac{(\mu x)^{r+j}}{(r+j)!}\bar{P}_{N-2,r,1}(o,s)$$

$$\left.+\sum_{j=0}^{m}\frac{(\mu x)^{j}}{(j)!}\bar{P}_{N-3,k-m+j,1}(o,s)\right].$$

...(4.2.65.)

Generalizing the results (4.2.57),(4.2.63) & (4.2.65), we get for n=N-l and r=k-m

$\bar{P}_{N-l,k-m,1}(x,s)$

$$=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}\left[\sum_{j=0}^{m}\sum_{r,r_1..r_{l-1}=1}^{k}\frac{(\mu x)^{r+r_1+..r_{l-1}+j}}{(r+r_1+..r_{l-1}+j)!}\bar{P}_{N,r,1}(o,s)\right.$$

$$+ \;............$$

$$+\sum_{j=0}^{m}\sum_{r=1}^{k}\frac{(\mu x)^{r+j}}{(r+j)!}\bar{P}_{N-l+1,r,1}(o,s)$$

$$\left.+\sum_{j=0}^{m}\frac{(\mu x)^{j}}{(j)!}\bar{P}_{N-l,k-m+1,1}(o,s)\right].$$

...(4.2.66.)

To obtain $\bar{P}_{n,1,1}(x,s)$, put N−l=n and m=k−1 in (4.2.66), we get

$$\bar{P}_{n,1,1}(x,s)$$

$$=e^{-(\mu+s)x}e^{-\int_0^x \eta(\mu)du}\left[\sum_{j=0}^{k-1}\sum_{r,r_1..r_{l-1}=1}^{k}\frac{(\mu x)^{r+r_1+..r_{l-1}+j}}{(r+r_1+..r_{l-1}+j)!}\bar{P}_{n+l,r,1}(o,s)\right.$$

$$+\sum_{j=0}^{k-1}\sum_{r=1}^{k}\frac{(\mu x)^{r+j}}{(r+j)!}\bar{P}_{n-2l,1}(o,s)$$

$$\left.+\sum_{j=0}^{k-1}\frac{(\mu x)^{j}}{(j)!}\bar{P}_{n,2,1}(o,s)\right].$$

...(4.2.67.)

In (4.2.38) we put n=N−b to get

$$\frac{\partial}{\partial x}\bar{P}_{N-b,k,1}(x,s)+\{s+\eta(x)\}\bar{P}_{n-b,k,1}(x,s) = \mu\bar{P}_{N,k,1}(x,s).$$

...(4.2.68.)

Solving (4.2.68) and using (4.2.48), we have

$$\bar{P}_{N-b,k,1}(x,s)=e^{-(\mu+s)x}e^{-\int_0^x \eta(\mu)du}\left[\mu x\bar{P}_{N,k,1}(o,s)+\bar{P}_{N-b,k,1}(o,s)\right].$$

...(4.2.69.)

Now corsider (4.2.39) taking n=N−b and r=k−1, we have

$$\frac{\partial}{\partial x}P_{N-b,k-1,1}(x,s)+\{s+\mu+\eta(x)\}P_{N-b,k-1,1}(x,s) = \mu P_{N-b,k,1}(x,s)$$

$$+\mu P_{N,k-1,1}(x,s)+\delta_{i,N-b}\delta_{m,k-1}\delta(x).$$

...(4.2.70.)

Solving (4.2.70) and using (4.2.48) & (4.2.69), we get

$$\bar{P}_{N-b,k-1,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}\left[\begin{array}{l}\frac{(\mu x)^2}{2!}\bar{P}_{N,k,1}(o,s) \\ +\mu x\bar{P}_{N-b,k,1}(o,s) \\ +\frac{(\mu x)^2}{2!}\bar{P}_{N,k,1}(o,s) \\ +\mu x\bar{P}_{N,k-1,1}(o,s) \\ +\bar{P}_{N-b,k-1,1}(o,s)\end{array}\right].$$

...(4.2.71.)

Proceeding similary, we get

$$\bar{P}_{N-2b,k-1,1}(x,s)=e^{-(\mu+s)x}\,e^{-\int_0^x \eta(\mu)du}\left[\begin{array}{l}\frac{(\mu x)^3}{3!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^2}{2!}\bar{P}_{N-b,k,1}(o,s)\\ +\mu x\bar{P}_{N-2b,k,1}(o,s)\\ +\frac{(\mu x)^3}{3!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^2}{2!}\bar{P}_{N-b,k,1}(o,s)\\ +\frac{(\mu x)^3}{3!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^2}{2!}\bar{P}_{N,k-1,1}(o,s)\\ +\mu x\bar{P}_{N-b,k-1,1}(o,s)\\ +\bar{P}_{N-2b,k-1,1}(o,s)\end{array}\right].$$

...(4.2.72.)

$$\bar{P}_{N-3b,k-1,1}(x,s)=\bar{e}^{(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du}\left[\begin{array}{l}\frac{(\mu x)^4}{4!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^3}{3!}\bar{P}_{N-b,k,1}(o,s)\\ +\frac{(\mu x)^2}{2!}\bar{P}_{N-2b,k,1}(o,s)\\ +\frac{(\mu x)^4}{4!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^3}{3!}\bar{P}_{N-b,k,1}(o,s)\\ +\frac{(\mu x)^4}{4!}\bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^3}{3!}\bar{P}_{N,k-1,1}(o,s)\\ +\frac{(\mu x)^2}{2!}\bar{P}_{N-b,k-1,1}(o,s)\\ +(\mu x)\bar{P}_{N-2b,k-1,1}(o,s\\ +\bar{P}_{N-3b,k-1,1}(o,s)\end{array}\right].$$

...(4.2.73.)

Generalizing the above results, we have

$$\bar{P}_{N-cb,k-1,1}(x,s)=e^{-(\mu+s)x}\,e^{-\int_0^x \eta(\mu)du}\left[\begin{array}{l}\frac{(\mu x)^{c+1}}{(c+1)!}\ \bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^{c}}{(c)!}\ \bar{P}_{N-b,k,1}(o,s)\\ +\frac{(\mu x)^{c-1}}{(c-1)!}\ \bar{P}_{N-2b,k,1}(o,s)\\ +\frac{(\mu x)^{c+1}}{(c+1)!}\ \bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^{c}}{(c)!}\ \bar{P}_{N-b,k,1}(o,s)\\ +\frac{(\mu x)^{c+1}}{(c+1)!}\ \bar{P}_{N,k,1}(o,s)\\ +\frac{(\mu x)^{c}}{(c)!}\ \bar{P}_{N,k-1,1}(o,s)\\ +\frac{(\mu x)^{c-1}}{(c-1)!}\ \bar{P}_{N-b,k-1,1}(o,s)\\ +\frac{(\mu x)^{c-2}}{(c-2)!}\ \bar{P}_{N,-2b,k-1,1}(o,s)\\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +\bar{P}_{N-cb,k-1,1}(o,s)\end{array}\right].$$

...(4.2.74.)

Now in (4.2.39) taking n=N-b and r=k-2, we have

$$\frac{\partial}{\partial x}\bar{P}_{N-b,k-2,1}(x,s)+\{s+\mu+\eta(x)\}\bar{P}_{N-b,k-2,1}(x,s)=\mu\bar{P}_{N-b,k-1,1}(x,s)+$$
$$+\mu\bar{P}_{N,k-2,1}(x,s)$$
$$+\delta_{i,N-b}\,\delta_{m,k-2}\,\delta(x).$$

...(4.2.75.)

Solving (4.2.75) and using (4.2.50) & (4.2.71), we get

$$\bar{P}_{N-b,k-2,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\frac{3(\mu x)^3}{3!}\bar{P}_{N,k,1}(o,s) + \frac{(\mu x)^2}{2!}\bar{P}_{N-b,k,1}(o,s) + \frac{2(\mu x)^2}{2!}\bar{P}_{N,k-1,1}(o,s) + \mu x\bar{P}_{N-b,k-1,1}(o,s) + \mu x\bar{P}_{N,k-2,1}(o,s) + \bar{P}_{N-b,k-2,1}(o,s) \right].$$

...(4.2.76.)

Proceeding similarly, we have

$$\bar{P}_{N-2b,k-2,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\frac{6(\mu x)^4}{4!}\bar{P}_{N,k,1}(o,s) + \frac{3(\mu x)^3}{3!}\bar{P}_{N-b,k,1}(o,s) + \frac{(\mu x)^2}{2!}\bar{P}_{N-2b,k,1}(o,s) + \frac{3(\mu x)^3}{3!}\bar{P}_{N,k-1,1}(o,s) + \frac{2(\mu x)^2}{2!}\bar{P}_{N-b,k-1,1}(o,s) + \mu x\bar{P}_{N-2b,k-1,1}(o,s) + \frac{(\mu x)^2}{2!}\bar{P}_{N,k-2,1}(o,s) + \mu x\bar{P}_{N-b,k-2,1}(o,s) + \bar{P}_{N-2b,k-2,1}(o,s) \right].$$

...(4.2.77.)

Generalizing the above results, we get

$$\bar{P}_{N-2b,k-m,1}(x,s)=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \Bigg[\frac{(\mu x)^m}{m!}\left[\frac{(\mu x)^2}{2!}\bar{P}_{N,k,1}(o,s) + \mu x\bar{P}_{N-b,k-1}(o,s) + \bar{P}_{N-2b,k,1}(o,s)\right]$$

$$+\frac{(\mu x)^{m-1}}{(m-1)!}\left[\bar{P}_{N,k-1,1}\frac{(\mu x)^2}{2!} + \mu x\bar{P}_{N-b,k-1,1}(o,s) + \bar{P}_{N-2,b,k-1,1}(o,s)\right]$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\mu x\left[\frac{(\mu x)^2}{2!}\bar{P}_{N,k-m+1}(O,s) + \mu x\bar{P}_{N-b,k-m+1,1}(o,s) + \bar{P}_{N-2b,k-m+1}(o,s)\right]$$

$$+\frac{(\mu x)^2}{2!}\bar{P}_{N,k-m,1}(o,s) + \mu x\bar{P}_{N-b,k-m,1}(o,s) + \bar{P}_{N-2b,k-m,1}(o,s)\Bigg]$$

...(4.2.78.)

Again generalizing the above result

$$\overline{P}_{N-cb,k-b,1}(x,s)$$

$$=e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\frac{(\mu x)^m}{m!} \left[\frac{(\mu x)^c}{c!}\overline{P}_{N,k,1}(o,s) + \frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{N-b,k,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!}\overline{P}_{N-2b,k,1}(o,s) + \ldots\ldots\ldots + \overline{P}_{N-cb,k,1}(o,s) \right] \right.$$

$$+\frac{(\mu x)^{m-1}}{(m-1)!}\left[\frac{(\mu x)^c}{c!}\overline{P}_{N,k-1,1}(o,s) + \frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{N-b,k-1,1}(o,s) + \ldots\ldots\ldots + \overline{P}_{N-cb,k-1,1}(o,s) \right]$$

$$+\mu x\left[\frac{(\mu x)^c}{c!}\overline{P}_{N,k-m+1}(o,s) + \frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{N-b,k-m+1,1}(o,s) + \ldots\ldots\ldots + \overline{P}_{N-cb,k-m+1,1}(o,s) \right]$$

$$\left. + \frac{(\mu x)^c}{c!}\overline{P}_{N,k-m,1}(o,s) + \frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{N-b,k-m,1}(o,s) + \ldots\ldots\ldots + \overline{P}_{N-cb,k-m,1}(o,s) \right].$$

...(4.2.79.)

In (4.2.79) put $n=N-1$ and $c=c-1$, we get

$$\bar{P}_{N-(c-1)b-1,k-m,1}(x,s)$$

$$= e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \Bigg[\frac{(\mu x)^m}{m!} \Bigg[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{N-1,k,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{N-1-b,k,1}(o,s) + \frac{(\mu x)^{c-3}}{(c-3)!} \bar{P}_{N-1-2b,k,1}(o,s) + \ldots\ldots + \bar{P}_{N-(c-1)b-1,k,1}(o,s) \Bigg]$$

$$+ \frac{(\mu x)^{m-1}}{(m-1)!} \Bigg[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{N-1,k-1,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{N-1-b,k-1,1}(o,s) + \ldots\ldots + \bar{P}_{N-(c-1)b-1,k-1,1}(o,s) \Bigg]$$

$$+ \ldots\ldots$$

$$+ (\mu x) \Bigg[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{N-1,k-m+1,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{N-b-1,k-m+1,1}(o,s) + \ldots\ldots + \bar{P}_{N-(c-1)b-1,k-m+1,1}(o,s) \Bigg]$$

$$+ \frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{N-1,k-m,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{N-b-1,k-m,1}(o,s) + \ldots\ldots + \bar{P}_{N-(c-1)b-1,k-m,1}(o,s) \Bigg].$$

...(4.2.80.)

Put $N-(c-1)b-1 = i$ in (4.2.80), we get

$$\bar{P}_{i,k-m,1}(x,s) =$$

$$e^{-(\mu+s)x}\, e^{-\int_0^x \eta(\mu)du} \left[\frac{(\mu x)^m}{m!} \left[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{i+(c-1)b,k,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{i+(c-2)b,k,1}(o,s) + \ldots\ldots + \bar{P}_{i,k,1}(o,s) \right] \right.$$

$$+ \frac{(\mu x)^{m-1}}{(m-1)!} \left[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{i+(c-1)b,k-1,1}(o,s) + \frac{(\mu x)^{c-2}}{(c-2)!} \bar{P}_{i+(c-2)b,k-1,1}(o,s + \ldots\ldots + \bar{P}_{i,k-1,1}(o,s) \right]$$

$$+ \ldots\ldots$$

$$+ (\mu x) \left[\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{i+(c-1)b,k-m+1.1}(o,s) + \ldots\ldots + \bar{P}_{i,k-m+1,1}(o,s) \right]$$

$$\left. + \frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{i+(c-1)b,k-m,1}(o,s) + \ldots\ldots + \bar{P}_{i,k-m,1}(o,s) \right].$$

...(4.2.81.)

To obtain $\overline{P}_{a,1,1}(x,s)$ put i=a and m=k-1 in (4.2.81),we get

$$
\begin{aligned}
&\overline{P}_{a,1,1}(x,s) = \\
&e^{-(\mu+s)x}\, e^{-\int_0^x \eta(u)du} \Bigg[\frac{(\mu x)^{k-1}}{(k-1)!}\Bigg[\frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{a+(c-1)b,k,1}(o,s) \\
&\qquad +\frac{(\mu x)^{c-2}}{(c-2)!}\overline{P}_{a+(c-2)b,k,1}(o,s) \\
&\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad +\overline{P}_{a,k,1}(o,s)\Bigg] \\
&+\frac{(\mu x)^{k-2}}{(k-2)!}\Bigg[\frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{a+(c-1)b,k-1,1}(o,s) \\
&\qquad +\frac{(\mu x)^{c-2}}{(c-2)!}\overline{P}_{a+(c-2)b,k-1,1}(o,s) \\
&\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad +\overline{P}_{a,k-1,1}(o,s)\Bigg] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+(\mu x)\Bigg[\frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{a+(c-1)b,2,1}(o,s) \\
&\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad +\overline{P}_{a,2,1}(o,s)\Bigg] \\
&+\frac{(\mu x)^{c-1}}{(c-1)!}\overline{P}_{a+(c-1)b,1,1}(o,s) \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+\overline{P}_{a,1,1}(o,s)\Bigg].
\end{aligned}
$$

...(4.2.82.)

To obtain $\bar{P}_{n,r,1}(x.s)$, put i=n and k-m=r in (4.2.81) we get

$$\bar{P}_{n,r,1}(x,s) =$$

$$e^{-(\mu+s)x}\, e^{-\int_0^x \eta(u)du} \left[\begin{array}{l} \frac{(\mu x)^{k-r}}{(k-r)!} \left[\begin{array}{l} \frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{n+(c-1)b,k,1}(o,s) \\ + \cdots\cdots\cdots \\ +\bar{P}_{n,k,1}(o,s) \end{array} \right] \\ +\frac{(\mu x)^{k-r-1}}{(k-r-1)!} \left[\begin{array}{l} \frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{n+(c-1)b,k-1,1}(o,s) \\ + \cdots\cdots\cdots \\ +\bar{P}_{n,k-1,1}(o,s) \end{array} \right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots \\ +(\mu x) \left[\begin{array}{l} \frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{n+(c-1)b,r+1.1}(o,s) \\ + \cdots\cdots\cdots \\ +\bar{P}_{n,r+1,1}(o,s) \end{array} \right] \\ +\frac{(\mu x)^{c-1}}{(c-1)!} \bar{P}_{n+(c-1)b,r,1}(o,s) \\ + \cdots\cdots\cdots\cdots\cdots\cdots \\ +\bar{P}_{n,r,1}(o,s) \end{array} \right].$$

...(4.2.83.)

CHAPTER 5

5.1 INTRODUCTION

In this chapter we study the queueing situation at a service facility where two types of customers,called priority and non-priority arrive. The service time distributions are negative exponential for both types of customers.Let μ_1 and μ_2 be the mean service rates of the priority and non-priority customers respectively. Scince the service time distributions are negative exponential, there is no differnce mathematically between the preemptive resume and preemptive repeat discipline because of the 'forgetfulness'property of the negative exponential distribution.thus our formulations are valid for both the rules.Use has been made of the supplementary variable technique and Laplace transforms. On inversion of the transforms,the busy period density function has been obtained for the general case,and particular cases of Poisson arrivals, and non-priority queue are cited.

We define a general bulk service rule on non-priority units that if there are only non-priority customers <b,then the entire queue will be taken up for the service and if ≥b then the server accepts the first 'b' non-priority customers for the service.

Let the customers arrivals be according to general disributions. Let $\eta(x)\Delta$ be the first order probability of the arrival of a customer in the interval $(x,x+\Delta)$ conditioned that no arrival has taken place up to a time x,so that if A(x) be the arrival time density have

$$A(x)=\eta(x)\exp\ [-\int_0^x \eta(a)du\ ].$$

Also let an arrival be a priority customer with a probability p and a non-priority customer with probability q, Where p+q=1,and let

$\eta_1(x)=p\eta(x)$,and $\eta_2(x)=q\eta(x)$,
So that $\eta_1(x)+\eta_2(x)=\eta(x)$.

Thus in the model studied here the two types of customers are assumed to emanate from the same source,and p and q reperesents the relative frequencies of occurance of the priority and non-priority customers in the source.

Now,let $P_{n,m}(x,t)dx$ be the probability that there are n priority and m non-priority customers present in the system at time t,the time sience the last arrival lyining between (x,x+dx),Also let there be limited waiting space for N priority and M non-priority units, so that when the waiting space is filled to its full capacity,any arriving unit whether priority or non-priority does not enter the system and is lost.

In 5.2 of this chapter we have analysed the busy period distribution of the system and the busy period density function and its Laplace transform are also obtained.In 5.3 the particular cases for Poisson arrivals and A non-priority queue are discussed.

5.2 BUSY PERIOD DISTRIBUTION

We have the following busy period equations for the systems.

$$\frac{\partial}{\partial t} P_{n,m}(x,t) + \frac{\partial}{\partial x} P_{n,m}(x,t) + [\mu_1+\eta(x)] P_{n,m}(x,t)$$

$$= \mu_1 P_{n+1,m}(x,t), \qquad \text{...(5.2.1.)}$$

$$\frac{\partial}{\partial t} P_{o,o}(x,t) + \frac{\partial}{\partial x} P_{o,o}(x,t) = \mu_1 P_{1,0}(x,t) + \mu_{2,} P_{0,1}(x,t),$$

$$\text{...(5.2.2.)}$$

$$\frac{\partial}{\partial t} P_{N,m}(x,t) + \frac{\partial}{\partial x} P_{N,m}(x,t) + [\mu_1+\eta(x)] P_{N,m}(x,t)=0,$$

$$\text{...(5.2.3.)}$$

$$\frac{\partial}{\partial t} P_{n,o}(x,t) + \frac{\partial}{\partial x} P_{n,o}(x,t) + [\mu_1+\eta(x)] P_{n,o}(x,t)$$

$$= \mu_1 P_{n+1,o}(x,t),$$

$$\text{...(5.2.4.)}$$

$$\frac{\partial}{\partial t} P_{N,O}(x,t) + \frac{\partial}{\partial x} P_{N,o}(x,t) + [\mu_1+\eta(x)] P_{N,o}(x,t)=0,$$

$$\text{...(5.2.5.)}$$

$$\frac{\partial}{\partial t} P_{o,M}(x,t) + \frac{\partial}{\partial x} P_{o,M}(x,t) + [\mu_2+\eta(x)] P_{o,M}(x,t)$$

$$= \mu_1 P_{1,M}(x,t), \qquad \text{...(5.2.6.)}$$

$$\frac{\partial}{\partial t} P_{N,M}(x,t) + \frac{\partial}{\partial x} P_{N,M}(x,t) + [\mu_1+\eta(x)]\, P_{N,M}(x,t)=0,$$

...(5.2.7.)

$$\frac{\partial}{\partial t} P_{o,m}(x,t) + \frac{\partial}{\partial x} P_{o,m}\ (x,t) + [\mu_2+\eta(x)]\, P_{o,m}(x,t)$$

$$= \mu_1 P_{1,m}(x,t) + \mu_2 P_{o,m+b}(x,t),$$ ...(5.2.8.)

$(1 < m < M-b)$

$$\frac{\partial}{\partial t} P_{o,m}(x,t) + \frac{\partial}{\partial x} P_{o,m}\ (x,t) + [\mu_2+\eta(x)]\, P_{o,m}(x,t)$$

$$= \mu_1 P_{1,m}(x,t)+ \mu_2 P_{o,m+1}(x,t),$$ ...(5.2.9.)

$(m-b+1<m\leq M)$

$$\frac{\partial}{\partial t} P_{n,M}(x,t) + \frac{\partial}{\partial x} P_{n,M}\ (x,t) + [\mu_1+\eta(x)]\, P_{n,M}(x,t)$$

$$=\mu_1 P_{n+1,M}(x,t).$$ ...(5.2.10.)

Subject to the following boundary conditions,

$$P_{n,m}(o,t)=\int_o^\infty P_{n-1,m}(x,t)\, \eta_1(x)dx + \int_o^\infty P_{n,m-1}(x,t)\eta_2\ (x)dx$$

...(5.2.11)

$$P_{o,o}(o,t) \;=\; 0, \qquad \ldots(5.2.12)$$

$$P_{N,m}(o,t) = \int_o^\infty [P_{N,m}(x,t) + P_{N-1,m}(x,t)]\eta_1(x)\,dx$$
$$+\int_o^\infty P_{N,m-1}(x,t)\eta_2(x)\,dx, \qquad \ldots(5.2.13)$$

$$P_{N,M}(o,t) = \int_o^\infty [P_{N,M}(x,t) + P_{N-1,M}(x,t)]\eta_1(x)\,dx$$
$$+\int_o P_{N,M-1}(x,t)\eta_2(x)\,dx,$$
$$\ldots(5.2.14)$$

$$P_{n,M}(o,t) = \int_o^\infty [P_{n,M}(x,t) + P_{n,M-1}(x,t)]\eta_2(x)\,dx$$
$$+\int_o^\infty P_{n-1,M}(x,t)\eta_1(x)\,dx,$$
$$\ldots(5.2.15)$$

$$P_{N,o}(o,t) = \int_o^\infty [P_{N,o}(x,t) + P_{N-1,o}(x,t)]\eta_1(x)dx,$$
$$\ldots(5.2.16)$$

$$P_{n,o}(o,t) = \int_o^\infty P_{n-1,o}(x,t)\,\eta_1(x)dx, \qquad \ldots(5.2.17)$$

$$P_{o,M}(o,t) = \int_o^\infty [P_{o,M}(x,t) + P_{o,M-1}(x,t)]\,\eta_2(x)dx,$$
$$\ldots(5.2.18)$$

$$P_{o,m}(o,t) = {}_{o}\int^{\infty} P_{o,m-1}(x,t)\,\eta_2(x)dx. \qquad \ldots(5.2.19)$$

Also the intial Condition is

$$P_{n,m}(x,o) = \delta_{1,n} . \delta_{o,m} . \delta(x), \qquad \ldots(5.2.20)$$

where $\delta(x)$ is the Dirac delta function and $\delta_{i,j}$ is the Kronecker delta :

$$\delta_{i,j} \begin{array}{l} = 1 \text{ for } i = j \\ = 0 \text{ for } i = j. \end{array} \qquad \ldots(5.2.21)$$

Now if $\bar{\gamma}(t)$,be the probaibility function of the busy period we have,

$$\gamma(t) = \frac{d}{dt} P_{o,o}(t). \qquad \ldots(5.2.22)$$

where

$$P_{o,o}(t) = {}_{o}\int^{\infty} P_{o,o}(x,t)\,dx, \qquad \ldots(5.2.23)$$

Let the Laplace transform of a funcation $F(t)$ be typically represented by $\bar{F}(s)$,so that

$$\bar{F}(s) = {}_{o}\int^{\infty} e^{-st} F(t)dt, \qquad \ldots(5.2.24)$$

we have

$$\bar{\gamma}(s) = S\,\bar{P}_{o,o}(s) = s\int_o^{\infty} \bar{P}_{o,o}(x,s)dx, \qquad \ldots(5.2.25)$$

Now taking Laplace transform of (5.2.1) - (5.2.10) we have

$$\frac{\partial}{\partial x}\bar{P}_{n,m}(x,s) + [s+\mu_1+\eta(x)]\,\bar{P}_{n,m}(x,s) = \mu_1\,\bar{P}_{n+1,m}(x,s),$$

$$\ldots(5.2.26)$$

$$\frac{\partial}{\partial x}\bar{P}_{o,o}(x,s) + s\,\bar{P}_{o,o}(x,s) = \mu_1\,\bar{P}_{1,o}(x,s) + \mu_2\,\bar{P}_{o,1}(x,s)$$

$$+\,\delta(x), \qquad \ldots(5.2.27)$$

$$\frac{\partial}{\partial x}\bar{P}_{N,m}(x,s) + [s+\mu_1+\eta(x)]\,\bar{P}_{N,m}(x,s) = 0, \qquad \ldots(5.2.28)$$

$$\frac{\partial}{\partial x}\bar{P}_{n.o}(x,s) + [s+\mu_1+\eta(x)]\,\bar{P}_{n,o}(x,s) = \mu_1\,\bar{P}_{n+1,o}(s,x),$$

$$(2 \le n \le N-1) \qquad \ldots(5.2.29)$$

$$\frac{\partial}{\partial x}P_{1,o}(x,s) + [s+\mu_1+\eta(x)]\,P_{1,o}(x,s) = \mu_1\,P_{2,o}(x,s)$$

$$+\,\delta_{1,n}\delta(x),$$

$$\ldots(5.2.30)$$

$$\frac{\partial}{\partial x}\overline{P}_{N,o}(x,s) + [s+\mu_1+\eta(x)]\,\overline{P}_{N,o}(x,s) = 0, \quad \ldots(5.2.31)$$

$$\frac{\partial}{\partial x}\overline{P}_{o,M}(x,s) + [s+\mu_2+\eta(x)]\,\overline{P}_{o,M}(x,s) = \mu_1\,\overline{P}_{1,M}(x,s),$$

$$\ldots(5.2.32)$$

$$\frac{\partial}{\partial x}\,\overline{P}_{N.M}(x,s) + [s+\mu_1+\eta(x)]\,\overline{P}_{N,M}(x,s) = 0, \quad \ldots(5.2.33)$$

$$\frac{\partial}{\partial x}\overline{P}_{o,m}(x,s) + [s+\mu_2+\eta(x)]\,\overline{P}_{o,m}(x,s) = \mu_1\,\overline{P}_{1,m}(x,s)$$

$$+\mu_2\,\overline{P}_{o,m+b}(x,s),$$

$$(1 < m < M-b) \qquad \ldots(5.2.34)$$

$$\frac{\partial}{\partial x}\overline{P}_{o,m}(x,s) + [s+\mu_2+\eta(x)]\,\overline{P}_{o,m}(x,s) = \mu_1\,\overline{P}_{1,m}(x,s)+\mu_2\overline{P}_{o,m+1},$$

$$(M-b+1 < m \leq M) \qquad \ldots(5.2.35)$$

$$\frac{\partial}{\partial x}\,\overline{P}_{n,M}(x,s) + [s+\mu_1+\eta(x)]\overline{P}_{n,M}(x,s) = \mu_1\,\overline{P}_{n+1,M}(x,s).$$

$$\ldots(5.2.36)$$

Now we have to determine $\bar{P}_{o,o}(x,s)$ from (5.2.27) for which we need $\bar{P}_{1,o}(x,s)$ and $\bar{P}_{o,1}(x,s)$.

For $\bar{P}_{1,o}(x,s)$, Sloving (5.2.31) we have

$$\bar{P}_{N,o}(x,s) = e^{-X_1} \bar{P}_{N,o}(o,s). \qquad \ldots(5.2.37)$$

where $X_1 = [(s+\mu_1)x + \int_o^x \eta(u)du]$

Now putting $n = N+1$ in (5.2.29), we get

$$\frac{\partial}{\partial x}\bar{P}_{N-1,o}(x,s) + [s+\mu_1+\eta(x)]\bar{P}_{N-1,o}(x,s)=\mu_1\bar{P}_{N,o}(x,s) \; . \qquad \ldots(5.2.38)$$

Solving (5.2.38) after substituting for $\bar{P}_{N,o}(x,s)$ from (5.2.37) we have

$$\bar{P}_{N-1,o}(x,s) = e^{-X_1} [\; \mu_1 x \; \bar{P}_{N,o}(x,s) + \bar{P}_{N-1,o}(o,s)]. \qquad \ldots(5.2.39)$$

Proceeding Similarly we have

$$\bar{P}_{2,o}(o,s) = \bar{e}^{X_1}\left[\frac{(\mu_1 \kappa)^{N-2}}{(N-2)!} \bar{P}_{N,o}(o,s) + \frac{(\mu_1 \kappa)^{N-3}}{(N-3)!} \bar{P}_{N-1,o}(o,s) + \ldots + \bar{P}_{2,o}(o,s) \right]$$

...(5.2.40)

And finally we have from (5.2.30) after substituting for

$\bar{P}_{2,o}(\kappa,s)$ from (5.2.40)

$$\bar{P}_{1,o}(\kappa,s) = \bar{e}^{X_1}\left[1 + \sum_{n=1}^{N} \frac{(\mu_1 \kappa)^{n-1}}{(n-1)!} \bar{P}_{n,o}(o,s) \right],$$

...(5.2.41)

Now ,for $\bar{P}_{o,1}(\kappa,s)$,solving (5.2.34) , we have

$$\bar{P}_{N,M}(\kappa,s) = \bar{e}^{X_1} \bar{P}_{N,M}(o,s). \qquad \ldots(5.2.42)$$

Substituting $n = N-1$ in (5.2.37) using (5.2.42) and then solving we have,

$$\bar{P}_{N-1.M}(\kappa,s) = e^{-X_1} [\mu_1 \kappa \bar{P}_{N,M}(o,s) + \bar{P}_{N-1,M}(\kappa,s)].$$

...(5.2.43)

Proceeding similarly ,we have

$$\bar{P}_{N-2,M}(x,s) = \bar{e}^{X_1}\left[\frac{(\mu_1 x)^2}{2!}\bar{P}_{N,M}(o,s) + \mu_1 x\,\bar{P}_{N-1,M}(o,s) + \bar{P}_{N-2,M}(o,s)\right].$$

...(5.2.44)

and finally , we have

$$\bar{P}_{1.M}(x,s) = e^{-X_1}\sum_{n=1}^{N}\frac{(\mu_1 x)^{n-1}}{(n-1)!}\bar{P}_{n,M}(o,s). \qquad ...(5.2.45)$$

Now substituting the value of $\bar{P}_{1,M}(x,s)$ from (5.2.45) in (5.2.32) and solving we get,

$$\bar{P}_{o,M}(x,s)= \bar{e}^{X_2}\left[\sum_{n=1}^{N}\frac{\bar{P}_{n,M}(o,s)}{(n-1)!}\int_o^x(\mu_1 y)^{n-1}e^{-(\mu_1-\mu_2)y}dy + \bar{P}_{o,N}(o,s)\right].$$

which can be furter simplified to

$$\bar{P}_{o,M}(x,s) = \bar{e}^{X_2}\left[\sum_{n=1}^{N}\frac{(\mu_1 x)^n}{(n)!h^n}\bar{P}_{n,M}(o,s)\,{}_1F_1(n;n+1;-hx) + \bar{P}_{o,M}(o,s)\right].$$

...(5.2.46)

where $X_2 = [(s+\mu_2)x + {}_o\int^x \eta(u)du]$ and $h = (\mu_1-\mu_2)$

Now ,(5.2.28) on taking for $m = M-b$ bcomes

$$\frac{\partial}{\partial x} \bar{P}_{N,M-b}(x,s) + [s+\mu_1+\eta(x)]\, \bar{P}_{N,M-b}(x,s) = 0 \quad \ldots(5.2.47)$$

Solving (5.2.47) ,we have

$$\bar{P}_{N,M-b}(x,s) = e^{-X_1}\, \bar{P}_{N,M-b}(o,s) \quad \ldots(5.2.48)$$

Proceeding similarly we have

$$\bar{P}_{N,M-2b}(x,s) = e^{-X_1}\, \bar{P}_{N,M-2b}(o,s), \quad \ldots(5.2.49)$$

and

$$\bar{P}_{N,M-3b}(x,s) = e^{-X_1}\, \bar{P}_{N,M-3b}(o,s). \quad \ldots(5.2.50)$$

Now (5.2.26) On taking for $n = N-1$ and $m=M-b$,becomes

$$\frac{\partial}{\partial x} \bar{P}_{N-1,M-b}(x,s)+[S+\mu_1+\eta(x)]\, \bar{P}_{N-1,M-b}(x,s)=\mu_1\bar{P}_{N,M-b}(x,s). \quad \ldots(5.2.51)$$

Substituting for $\bar{P}_{N,M-b}(x,s)$ from (5.2.48) and then solving (5.2.51) we have,

$$\bar{P}_{N-1,M-b}(x,s)=e^{-X_1}\ [\mu_1 x\ \bar{P}_{N,M-b}(o,s)+\bar{P}_{N-1,M-b}(o,s)].$$

...(5.2.52)

Proceeding ,Similarly we have

$$\bar{P}_{N-2,M-b}(x,s) = e^{-X_1}\left[\frac{(\mu_1 x)^2}{2\ !}\bar{P}_{N,M-b}(o,s) + \mu_1 x\ \bar{P}_{N-1,M-b}(o,s) + \bar{P}_{N-2,M-b}(o,s)\right],$$

...(5.2.53.)

$$\bar{P}_{N-3,M-b}(x,s) = \bar{e}^{X_1}\left[\frac{(\mu_1 x)^3}{3!}\bar{P}_{N,M-b}(o,s) + \frac{(\mu_1 x)^2}{2!}\ \bar{P}_{N-1,M-b}(o,s) + \mu_1\ x\ \bar{P}_{N-2,M-b}(o,s) + \bar{P}_{N-3,M-b}(o,s)\right],$$

...(5.2.54)

and finally we have

$$\bar{P}_{1,M-b}\ (x,s) = \bar{e}^{X_1}\left[\frac{(\mu_1 x)^{N-1}}{(N-1)!}\bar{P}_{N,M-b}(o,s) + \frac{(\mu_1 x)^{N-2}}{(N-2)!}\ \bar{P}_{N-1,M-b}(o,s) + \frac{(\mu_1 x)^{N-3}}{(N-3)!}\ \bar{P}_{N-2,M-b}(o,s) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \bar{P}_{1,M-b}(o,s)\right].$$

...(5.2.55)

Now (5.2.34) on taking for m=M-b becomes,

$$\frac{\partial}{\partial x}\bar{P}_{o,M-b}(x,s) + [s+\mu_2+\eta(x)]\bar{P}_{o,M-b}(x,s)=\mu_1\bar{P}_{1,M-b}(x,s)$$
$$+\mu_2\bar{P}_{o,M}(o,s).$$
$$(1 < m \leq M-b) \qquad \ldots(5.2.56)$$

Now substituting for $\bar{P}_{1,M-b}(x,s)$ and $\bar{P}_{o,M}(x,s)$ in (5.2.56) from (5.2.55) and (5.2.46) respectively , and then solving we get,

$$\bar{P}_{o,M-b}(o,s)=e^{-X_2}\left[\sum_{n=o}^{N}\bar{P}_{n,M-b}(o,s)\frac{(\mu_1 x)^n}{n!h^n}\,{}_1F_1(n;n+1;-hx)\right.$$
$$\left.+\mu_2 x\sum_{n=o}^{N}\bar{P}_{n,M}(o,s)\frac{(\mu_1 x)^n}{(n+1)!h^n}\,{}_1F_1(n;n+2;-hx)\right]$$

$$(1 < m < M-b) \qquad \ldots(5.2.57)$$

Now (5.2.26) for n = N-1 and m = M-2b becomes

$$\frac{\partial}{\partial x}\bar{P}_{N-1,M-2b}(x,s)+[s+\mu_1+\eta(x)]\bar{P}_{N-1,M-2b}(x,s)=\mu_1\bar{P}_{N,M-2b}(x,s).$$
$$\ldots(5.2.58)$$

Substituting for $\bar{P}_{N,M-2b}(x,s)$ from (5.2.49) in (5.2.57) and then solving we have

$$\bar{P}_{N-1,M-2b}(x,s) = e^{-X_1} [\mu_1 x \bar{P}_{N,M-2b}(x,s) + \bar{P}_{N-1,M-2b}(x,s)]. \quad \ldots (5.2.59)$$

Proceeding similarly , we have

$$\bar{P}_{N-1,M-3b}(x,s) = e^{-X_1} [\mu_1 x \bar{P}_{N,M-3b}(o,s) + \bar{P}_{N-1,M-3b}(o,s)]. \quad \ldots (5.2.60)$$

Now (5.2.26) for n = N-2 and m=M-2b becomes,

$$\frac{\partial}{\partial x} \bar{P}_{N-2,M-2b}(x,s) + [s+\mu_1+\eta(x)] \bar{P}_{N-2,M-2b}(x,s) = \mu_1 \bar{P}_{N-1,M-2b}(x,s). \quad \ldots (5.2.61)$$

Substituting for $\bar{P}_{N-1,M-2b}(x,s)$ from (5.2.58) in (5.2.60) and then solving we have,

$$\bar{P}_{N-2,M-2b}(x,s) = \bar{e}^{X_1} \left[\frac{(\mu_1 x)^2}{2!} \bar{P}_{N,M-2b}(o,s) + \mu_1 x \bar{P}_{N-1,M-2b}(o,s) + \bar{P}_{N-2,M-2b}(o,s) \right]. \quad \ldots (5.2.62)$$

Proceeding similary , we get

$$\bar{P}_{N-3,M-2b}(\kappa,s) = e^{-X_1}\left[\frac{(\mu_1\kappa)^3}{3!}\bar{P}_{N,M-2b}(o,s)+\frac{(\mu_1\kappa)^2}{2!}\cdot \bar{P}_{N-1,M-2b}(o,s)+\mu_1\kappa\,\bar{P}_{N-2,M-2b}(o,s) + P_{N-3,M-2b}(o,s)\right],$$

...(5.2.63)

and finally , we have

$$\bar{P}_{1,m-2b}(\kappa,s) = e^{-X_1}\left[\frac{(\mu_1\kappa)^{N-1}}{(N-1)!}\bar{P}_{N,M-2b}(o,s)+\frac{(\mu_1\kappa)^{N-2}}{(N-2)!}\cdot \bar{P}_{N-1,M-2b}(o,s)+\frac{(\mu_1\kappa)^{N-3}}{(N-3)!}\cdot \bar{P}_{N-2,M-2b}(o,s)+\ldots.+\bar{P}_{1,M-3b}(o,s)\right].$$

...(5.2.64)

Proceeding similary , we have

$$\bar{P}_{1,M-3b}(\kappa,s) = e^{-X_1}\left[\frac{(\mu_1\kappa)^{N-1}}{(N-1)!}\bar{P}_{N,M-3b}(o,s)+\frac{(\mu_1\kappa)^{N-2}}{(N-2)!}\cdot \bar{P}_{N-1,M-3b}(o,s)+\frac{(\mu_1\kappa)^{N-3}}{(N-3)!}\cdot \bar{P}_{N-2,M-3b}(o,s)+\ldots+\bar{P}_{1,M-3b}(o,s)\right].$$

...(5.2.65)

Now (5.2.44) for m = M-2b becomes,

$$\frac{\partial}{\partial x}\bar{P}_{o,M-2b}(x,s) + [s+\mu_1+\eta(x)]\bar{P}_{o,M-2b}(x,s)=\mu_1\,\bar{P}_{1,M-2b}(x,s) + \mu_2\bar{P}_{o,M-b}(x,s).$$

$$(1 < m < M-b) \qquad \ldots(5.2.66)$$

Substituting for $\bar{P}_{o,M-2b}(x,s)$ and $\bar{P}_{o,M-b}(x,s)$ from (5.2.64) and (5.2.57) respectively in (5.2.66) and then solving , we get

$$
\begin{aligned}
\bar{P}_{o,M-2b}(\kappa,s) = e^{-X_2} \Bigg[& \bar{P}_{o,M-2b}(o,s) + \bar{P}_{N,M-2b}(o,s) \frac{(\mu_1\kappa)^N}{N!\ h^n} \\
& {}_1F_1\ (N;N+1;-h\kappa) + \bar{P}_{N-1,M-2b}(o,s) \\
& \frac{(\mu_1\kappa)^{N-1}}{(N-1)!h^{N-1}}\ {}_1F_1\ (N-1;N;-h\kappa) + \ldots\ldots \\
& + \frac{\mu_1\kappa}{h}\ \bar{P}_{1,M-2b}(o,s)\ {}_1F_1\ (1;2;-h\kappa) \\
& + \mu_2\kappa\ \bar{P}_{N,M-b}(o,s)\ \frac{(\mu_1\kappa)^N}{(N+1)!h^N} \cdot \\
& {}_1F_1\ (N;N+2;-h\kappa) + \mu_2\kappa\bar{P}_{N-1,M-b}(o,s) \\
& \frac{(\mu_1\kappa)^{N-1}}{N!\ h^{N-1}}\ {}_1F_1\ (N-1;N+1;-h\kappa) \\
& + \mu_2\kappa\ \bar{P}_{N-2,M-b}(o,s)\ \frac{(\mu_1\kappa)^{N-2}}{(N-1)!h^{N-2}} \cdot \\
& {}_1F_1(N-2;N;-h\kappa) + \ldots + \frac{\mu_1\mu_2}{h}\ \frac{X^2}{2!} \\
& \bar{P}_{1,M-b}(o,s)\ {}_1F_1(1;3;-h\kappa) + \frac{(\mu_2\kappa)^2}{2!} \\
& \bar{P}_{o,M}(o,s) + \mu_2\kappa\ \bar{P}_{c\ M-b}(o,s) + (\mu_2\kappa)^2 \cdot \\
& \sum_{n=1}^{N} \bar{P}_{n,M}(o,s)\ \frac{(\mu_1\kappa)^n}{h^n\ (n+2)!} \cdot \\
& {}_1F_1(n;n+3;-h\kappa) \Bigg]
\end{aligned}
$$

...(5.2.66)

Which on further simplification, gives :

$$\bar{P}_{o,M-2b}(\kappa,s) = \bar{e}^{X_2} \left[\sum_{n=o}^{N} \bar{P}_{n,M-2b}(o,s) \frac{(\mu_1\kappa)^n}{n!h^n} \; {}_1F_1(n;n+1;-h\kappa) + \mu_2\kappa \sum_{n=o}^{N} \frac{(\mu_1\kappa)^n}{(n+1)!h^n} \; {}_1F_1 \; (n;n+2;-h\kappa) \,. \; \bar{P}_{n,M-b}(o,s) + (\mu_2\kappa)^2 \sum_{n=o}^{N} \bar{P}_{n,M}(o,s) \,. \; \frac{(\mu_1\kappa)^n}{(n+2)!h^n} \; {}_1F_1(n;n+3;-h\kappa) \right],$$

...(5.2.67)

Proceeding similary we have :

$$\bar{P}_{o,M-3b}(o,s) = \bar{e}^{X_2} \left[\sum_{n=o}^{N} \bar{P}_{n,M-3b}(o,s) \frac{(\mu_1\kappa)^n}{n!\; h^n} \; {}_1F_1(n;n+1;-h\kappa) + \mu_2\kappa \sum_{n=o}^{N} \bar{P}_{n,M-2b}(o,s) \frac{(\mu_1\kappa)^n}{(n+1)!h^n} \; {}_1F_1(n;n+2 \; -h\kappa) + (\mu_2\kappa)^2 \sum_{n=o}^{N} \bar{P}_{n,M-b}(o,s) \frac{(\mu_1\kappa)^n}{(n+2)!h^n} \,. \; {}_1F_1(n;n+3;-h\kappa) + (\mu_2\kappa)^3 \sum_{n=o}^{N} \bar{P}_{n,M}(o,s) \; \frac{(\mu_1\kappa)^n}{(n+3)!h^n} \; {}_1F_1(n;n+4);-h\kappa) \right].$$

...(5.2.68)

Generalizing the results (5.2.57),(5.2.67),(5.2.68) we have

$$\bar{P}_{o,M-(c-1)b}(x,s) = \bar{e}^{X_2} \Bigg[\sum_{n=o}^{N} \bar{P}_{n,M-(c-1)b}(o,s) \frac{(\mu_1 x)^n}{n!h^n} \cdot {}_1F_1(n;n+1;-hx) + \mu_2 x \sum_{n=o}^{N} \bar{P}_{n,M-(c-2)b}(o,s) \frac{(\mu_1 x)^n}{(n+1)!h^n} \cdot {}_1F_1(n;n+2;-hx) + (\mu_2 x)^2 \sum_{n=o}^{N} \bar{P}_{n,M-(c-3)b}(o,s) \frac{(\mu_1 x)^n}{(n+2)!h^n} {}_1F_1(n;n+3;-hx) + \ldots\ldots\ldots\ldots + (\mu_2 x)^{c-1} \sum_{n=o}^{N} \bar{P}_{n,m}(o,s) \frac{(\mu_1 x)^n}{h^n (n+c-1)!} \cdot {}_1F_1(n;n+c;-hx) \Bigg].$$

...(5.2.69)

Putting $M-(c-1)b = i$ in (5.2.69) we have,

$$\bar{P}_{o,i}(\kappa,s) = e^{-X_2} \Bigg[\sum_{n=o}^{N} \bar{P}_{n,i}(o,s) \frac{(\mu_1\kappa)^n}{(n+1)!h^n} \cdot {}_1F_1(n;n+1;-h\kappa) + \mu_2\kappa \sum_{n=o}^{N} \bar{P}_{n,i+b}(o,s) \frac{(\mu_1\kappa)^n}{(n+2)!h^n} {}_1F_1(n;n+2;-h\kappa) + (\mu_2\kappa)^2 \sum_{n=o}^{N} \bar{P}_{n,i+2b}(o,s) \frac{(\mu_1\kappa)^n}{(n+3)!h^n} {}_1F_1(n;n+3;-h\kappa) + \ldots\ldots\ldots\ldots + (\mu_2\kappa)^{(M-i)/b} \sum_{n=o}^{N} \bar{P}_{n,M}(o,s) \frac{(\mu_1\kappa)^n}{h^n(1+n+\frac{M-i}{b})} {}_1F_1(n;1+n+\frac{M-i}{b};h\kappa) \Bigg].$$

...(5.2.70)

Put i = 1 in (5.2.70),we hvae:

$$\bar{P}_{o,1}(x,s) = e^{-X_2} \Bigg[\sum_{n=o}^{N} \bar{P}_{n,1}(o,s) \frac{(\mu_1 x)^n}{(n+1)!h^n} \cdot {}_1F_1(n;n+1;-hx) + \mu_2 x \sum_{n=o}^{N} \bar{P}_{n,1+b}(o,s) \frac{(\mu_1 x)^n}{(n+2)!h^n} \cdot {}_1F_1(n;n+2;-hx) + (\mu_2 x)^2 \sum_{n=o}^{N} \bar{P}_{n,1+2b}(o,s) \frac{(\mu_1 x)^n}{(n+3)!h^n} \cdot {}_1F_1(n;n+3;-hx) + \ldots\ldots\ldots\ldots + (\mu_2 x)^{(M-1)/b} \sum_{n=o}^{N} \bar{P}_{n,M}(o,s) \frac{(\mu_1 x)^n}{h^n[1+n+(m-1)/b]!} {}_1F_1(n;1+n+\frac{m-1}{b};hx) \Bigg].$$

...(5.2.71)

Now,substituting for $P_{1,o}(x,s)$ and $P_{o,1}(x,s)$ from (5.2.41) and (5.2.71) in (5.2.37) and solving.we have

$$\bar{P}_{o,o}(x,s) = e^{-sx} \Bigg[\mu_1 \int e^{sx} \cdot \bar{e}^{X_1} \Big[1 + \sum_{n=1}^{N} \frac{(\mu_1 x)^{n-1}}{(n-1)!} \bar{P}_{n,o}(o,s) \Big] dx + \mu_2 \int e^{sx} \cdot \bar{e}^{X_2} [\bar{P}_{o,1}(o,s)] dx \Bigg].$$

Which can be funther simplified to

$$\bar{P}_{o,o}(x,s) = e^{-sx}\left[\sum_{n=1}^{N} \frac{(\mu_1)^n}{(n-1)!}\bar{P}_{n,o}(o,s)\int_o^x e^{-y_1} y^{n-1}\,dy + \mu_1 \int_o^x e^{-y_1}\,dy + \sum_{m=1}^{1+(M-1)/b} \frac{\mu_2^m \bar{P}_{o,m}(o,s)}{m\,!} \int_o^x y^{m-1} e^{-y_2}\,dy + \sum_{m=1}^{1+(M-1)/b} \sum_{n=1}^{N} \bar{P}_{n,M}(o,s) \frac{\mu_1^n \mu_2^m}{(n+m)!h^n} \int_o^x y^{n+m-1}\,{}_1F_1(n;n+m;-hy)\,dy\right].$$

...(5.2.72)

where $y_1 = [\mu_1 y + \int_o^y \eta(u)du]$

$y_2 = [\mu_2 y + \int_o^y \eta(u)du]$

Thus,substituing (5.2.72) in (5.2.35) and changing the order of intergation and simplying , we have :

$$\bar{\gamma}(s) = \sum_{n=1}^{N} \frac{\mu_1^n}{(n-1)!} \bar{P}_{n,o}(o,s) \int_o^{\infty} [e^{-(s+\mu_1)y}\, y^{n-1}\, e^{-\int_o^y \eta(u)du}]dy$$

$$+ \mu_1 \int_o^{\infty} [e^{|-(s+\mu_1)y|}\, e^{\int_o^y \eta(u)du}]dy$$

$$+ \sum_{m=1}^{1+(M-1)/b} \frac{\mu_2^m \bar{P}_{o,m}(o,s)}{\Gamma m} \int_o^{\infty} \begin{bmatrix} e^{-(s+\mu_2)y}\, y^{m-1}. \\ e^{-\int_o^y \eta(u)du} \end{bmatrix} dy$$

$$+ \sum_{m=1}^{1+(M-1)/b} \sum_{n=1}^{N} \frac{\mu_1^n \mu_2^m \bar{P}_{n,m}(o,s)}{\Gamma(n+m)\, h^n}$$

$$\int_o^{\infty} [e^{-(s+\mu_2)y}\, y^{n+m-1}\, {}_1F_1(n;n+m;-hy)\, e^{-\int_o^y \eta(u)du}]\, dy .$$

...(5.2.73)

Now on inversion (5.2.73) gives the busy period density function :

$$\gamma(t) = \left[\sum_{n=1}^{N} \frac{\mu_1^n P_{n,o}(o,t)}{(n-1)!} * t^{n-1}\, e^{(-\mu_1 t - \int_o^t \eta(u)du)} + \mu_1 e^{(-\mu_1 t - \int_o^t \eta(u)du)} + \sum_{m=1}^{1+(M-1)/b} \frac{\mu_2^m P_{o,m}(o,t)}{(m-1)!} * \sum_{m=1}^{1+(M-1)/b} \sum_{n=1}^{N} \frac{\mu_1^n \mu_2^m}{(n+m-1)!h^n} P_{n,m}(o,t) * t^{n+m-1}\, e^{(-\mu_2 t - \int_o^t \eta(u)du}\, {}_1F_1(n;n+m;-hx) \right].$$

...(5.2.74)

where, * stands for convolution.

Now $\overline{P}_{n,m}(o,t)$ $(0 \leq n \leq N,\ 0 \leq m \leq M)$ can be determined from the boundary conditions (5.2.21) - (5.2.29) which completes the determination of the busy period dnsity function.

<u>REMARK</u> :- The evaluation of the $P_{n,m}(o,t)$ is quite involed but can be handled by a computer . For instance in the case N=M=2 the problem requires inversion of an 8x8 matriv.

5.3 PARTICULAR CASES

(i) POISSON ARRIVALS :-Resultforthiscasecan begotbytaking $\eta x = \lambda$,the mean arrival rate . Thus , we have:

$$\gamma(t) = \left[\sum_{n=1}^{N} \frac{\mu_1^n P_{n,o}(o,t)}{(n-1)!} * t^{n-1} \quad e^{-(\lambda+\mu_1)t} + \mu_1 \; e^{-(\lambda+\mu_1)t} \right.$$

$$+ \sum_{m=1}^{1+(M-1)/b} \frac{\mu_2^m P_{o,m}(o,t)}{(m-1)!} * t^{m-1} \quad e^{-(\lambda+\mu_2)t}$$

$$+ \sum_{m=1}^{1+(M-1)/b} \sum_{n=1}^{N} \frac{\mu_1^n \mu_2^m}{(n+m-1)!h^n} P_{n,m}(o,t) * t^{n+m-1}$$

$$\left. e^{-(\lambda+\mu_2)t} \quad {}_1F_1(n;n+mi;-ht) \right].$$

...(5.3.1)

(ii)A NON-PRIORITY QUEUE !-Herne we set $1+(M-1)/b = o, \mu_2 = 0, \mu_1 = \mu$ and obtain from (5.2.73) and (5.2.74)

$$\gamma(S) = \left[\sum_{n=1}^{N} \frac{\mu^n}{(n-1)!} \bar{P}_{n,o}(o,s) \int_o^{\infty} [e^{-(s+\mu)y} \; y^{n-1} \; e^{-\int_o^y \eta(u)du}]dy \right.$$

$$\left. + \mu \int_o^{\infty} [e^{-(s+\mu)y} \; e^{-\int_o^y \eta(u)du}] \, dy \right].$$

...(5.3.2)

$$\gamma(t) = \left[\sum_{n=1}^{N} \frac{\mu^n P_{n,o}(t)}{(n-1)!} * t^{n-1} \; e^{(-\mu t - \int_o^t \eta(u)du)} + \mu \, e^{(-\mu t - \int_o^t \eta(u)du)} \right]. \qquad \ldots(5.3.3)$$

CHAPTER 6

6.1 INTRODUCTION

The presant work is the modification of preemptive priority queue , studied by R.Sivasamy[178].We consider a single server preemptive priority queueing system consisting of two types of units, with unlimited Poisson inputs and exponential service time distributions. The higher priority units are served in batchs according to a general bulk service rule and they have preemptive priority over lower priority units . The transient state behaviour of the model is discussed and queue length distributions,stability condition and the mean queue lengths are obtained.

Queues with more then one type of input and with different priority rules have been solved under various assumptions .Heatheote [87] has studied a preemptive priority queueing problem with two priorities. Sivasamy [178] has analysed a non preemptive priority queue by introducing a general bulk service rule in the lower priority queue. In this chapter,we apply similar ideas as in Heathecote[87] and Sivasamy[178], to study a preemptive priority queueing problem byintroducting the general bulk service rule in the top priority queue .

Let us consider a single server facility in which two independent Poisson classes of units, to be called type-1 units (top priority or higher priority units) and types-2 units (lower priority or non priority units) arrives at rates λ_1 and λ_2 respectively and form separate queues. The capacity of types-1 units may be at the most M and that of type-2 units may be unlimited . Let the service times of the type-1 and type-2 units be independently

exponentially distributed random variables with means μ_1^{-1} and μ_2^{-1} respectively .The type-1 units have preemptive priority over the type-2 units and it operates as folows :

As soon as a type-1 units arrives the server breakes his service to a type-2 unit if any, and remains idle.Then the idle server opens service for type-1 units when the queue lungth reaches size 'a' units . If, at a service epoch, the server finds 'q' types-1 units in the system where $1 \leq q \leq a-1$, no matter how many type-2 units are present, he waits until there accumulates 'a' type-1 units, whereupon he removes the batch of 'a' units for service if he finds more then 'b' units, and at the most M units he picks a batch of 'b' units at random for service, while other units wait in the queue .If none of the type-1 units is present at an epoch, the server starts servicing the type-2 units . Under this preemptive priority discipline a type-2 unit may be preempted any number of times.

This queueing system, in a real life situation may be observed in the following taxidriver cum repairman problem . The server is a taxidriver who is also attending to the service of automobilis if he is free from taxi driving , while he is at serivce as a repairman , if there is a demand for his taxi then he abandons the automobilies service but wait till a minimum of 'a' customers .Then at the subsequent service epochs, the server attends these two different jobs as indicated in the discription of the queueing system under study.

In section 6.1, we have obtained the transient state behaviour of the system. In 6.2, the mean queue lengths for higher priority as well as lower priority customers are obtained.

6.2 THE TRANSIENT STATE BEHAVIOUR

We define :

$P_{m,n}(t)$: The probability that there are $m(0\le m\le M)$, type-1 units and n ($n\ge 0$) type-2units present in the system . Obviously the server will be at restwhen $(1\le m\le a-1, n\ge 0)$ and the server will be busy with the type-2units when $(m=0, n\ge 1)$.

$Q_{m,n}(t)$: The probability that the server is busy with a batch of type-1 units and there are m types-1 units (excluding the type-1units included in the batch under service)and n type-2 units wait in the respecitve queues. It is non zero for all $m,n\ge 0$

The transition equation of the queueing model under study are given by :

$$P_{o,o}(t+\Delta t) = [1-(\lambda_1+\lambda_2)\Delta t]P_{o,o}(t)+\mu_2 \Delta t\, P_{o,1}(t)+\mu_1\Delta t\, Q_{o,o}, \quad \ldots(6.2.1)$$

$$P_{o,n}(t+\Delta t) = [1-(\lambda_1+\lambda_2+\mu_2)\Delta t]P_{o,n}(t)+\lambda_2 \Delta t\, P_{o,n-1}(t)$$

$$+\mu_2 \Delta t\, P_{o,n+1}(t) +\mu_1 \Delta t\, Q_{o,n}(t),$$

$$(n\ge 1) \qquad \ldots(6.2.2)$$

$$P_{m,o}(t+\Delta t)=[1-(\lambda_1+\lambda_2)\Delta t]P_{m,o}(t)+\lambda_1\Delta t\, P_{m-1,o}(t)+\mu_1\Delta t\, Q_{m,o}(t), \quad \ldots(6.2.3)$$

$$(1 \le m \le a-1)$$

$$P_{m,n}(t+\Delta t) = [1-(\lambda_1+\lambda_2)\Delta t]P_{m,n}(t)+\lambda_1 \Delta t\, P_{m-1,n}(t)$$

$$+\lambda_2\Delta t\, P_{m,n-1}(t)+\mu_1 \Delta t\, Q_{m,n}(t),$$

$$(1 \le m \le a-1 \;;\; n \ge 1) \qquad \ldots(6.2.4)$$

$$Q_{o,o}(t+\Delta t) = [1-(\lambda_1+\lambda_2+\mu_1)\Delta t]Q_{o,o}(t)+\lambda_1 \Delta t\, P_{a-1,o}(t)$$

$$+ \mu_1\Delta t \sum_{k=a}^{b} Q_{k,o}(t), \qquad \ldots(6.2.5)$$

$$Q_{o,n}(t+\Delta t) = [1-(\lambda_1+\lambda_2+\mu_1)\Delta t]Q_{o,n}(t)+\lambda_1 \Delta t\, P_{a-1,n}(t)$$

$$+\lambda_2\Delta t\, Q_{o,n-1}(t) +\mu_1 \Delta t \sum_{k=a}^{b} Q_{k,n}(t),$$

$$(n\ge 1) \qquad \ldots(6.2.6)$$

$$Q_{m,o}(t+\Delta t) = [1-(\lambda_1+\lambda_2+\mu_1)\Delta t]Q_{m,o}(t)+\lambda_1 \Delta t\, Q_{m-1,o}(t)$$

$$+\mu_1 \Delta t\, Q_{m+b,o}(t),$$

$$(1 \le m < M-b+1) \qquad \ldots(6.2.7)$$

$$Q_{m,n}(t+\Delta t) = [1-(\lambda_1+\lambda_2+\mu_1)\Delta t]Q_{m,n}(t)+\lambda_1 \Delta t\, Q_{m-1,n}(t)$$

$$+\mu_1 \Delta t\, Q_{m+b,n}(t)+\lambda_2 \Delta t\, Q_{m,n-1}(t),$$

$$(1 \le m < m-b+1 \;;\; n \ge 1) \ldots(6.2.8)$$

$$Q_{m,o}(t)\ (t+\Delta t) = [1-\mu_1\Delta t)\ Q_{m,o}(t) + \lambda_1\ \Delta t\ Q_{m-1,o}(t),$$

$$\ldots(6.2.9)$$

$$Q_{m,n}(t)\ (t+\Delta t) = [1-\mu_1\Delta t)\ Q_{m,n}(t) + \lambda_1\ \Delta t\ Q_{m-1,n}(t)$$

$$+\lambda_2\ \Delta t\ Q_{m,n-1}(t). \qquad \ldots(6.2.10)$$

$$(M-b+1 \leq m \leq M\ ;\ n \geq 1)$$

Transposing and letting $\Delta t \to 0$, the above equations redue in the following from :

$$\frac{d}{dt} P_{o,o}(t) + (\lambda_1 + \lambda_2) P_{o,o}(t) - \mu_2\ P_{o,1}(t) - \mu_1\ Q_{o,o}(t) = 0\ ,$$

$$\ldots(6.2.11)$$

$$\frac{d}{dt}\ P_{o,n}(t) + (\lambda_1 + \lambda_2 + \mu_2) P_{o,n}(t) - \lambda_2\ P_{o,n-1}(t) - \mu_2\ P_{o,n+1}(t)$$

$$-\ \mu_1 Q_{o,n}(t) = 0\ ,$$

$$(n \geq 1) \qquad \ldots(6.2.12)$$

$$\frac{d}{dt}\ P_{m,o}(t) + (\lambda_1 + \lambda_2) P_{m,o}(t) - \lambda_1\ P_{m-1,o}(t) - \mu_1\ Q_{m,o}(t) = 0,$$

$$(1 \leq m \leq a-1) \qquad \ldots(6.2.13)$$

$$\frac{d}{dt} P_{m,n}(t)+(\lambda_1+\lambda_2)P_{m,n}(t)-\lambda_1 P_{m-1,n}(t)-\lambda_2 P_{m,n-1}(t)$$

$$-\mu_1 Q_{m,n}(t) = 0, \qquad \ldots(6.2.14)$$

$$(1 \leq m \leq a-1 \; ; \; n \geq 1)$$

$$\frac{d}{dt} Q_{o,o}(t)+(\lambda_1+\lambda_2+\mu_1)Q_{o,o}(t)-\lambda_1 P_{a-1,o}(t)-\mu_1 \sum_{k=a}^{b} Q_{k,o}(t)=0,$$

$$\ldots(6.2.15)$$

$$\frac{d}{dt} Q_{o,n}(t)+(\lambda_1+\lambda_2+\mu_1)Q_{o,n}(t)-\lambda_1 P_{a-1,o}(t)-\lambda_2 Q_{o,n-1}(t)$$

$$-\mu_1 \sum_{k=a}^{b} Q_{k,n}(t) = 0, \qquad \ldots(6.2.16)$$

$$(n\geq 1)$$

$$\frac{d}{dt} Q_{m,o}(t)+(\lambda_1+\lambda_2+\mu_1)Q_{m,o}(t)-\lambda_1 Q_{m-1,o}(t)-\mu_1 Q_{m+b,o}(t)=0,$$

$$(1 \leq m < M-b+1) \qquad \ldots(6.2.17)$$

$$\frac{d}{dt} Q_{m,n}(t)+(\lambda_1+\lambda_2+\mu_1)Q_{m,n}(t)-\lambda_1 Q_{m-1,n}(t)-\mu_1 Q_{m+b,n}(t)=0,$$

$$\ldots(6.2.18)$$

$$\frac{d}{dt} Q_{m,o}(t) + \mu_1 Q_{m,o}(t) - \lambda_1 Q_{m-1,o}(t) = 0, \qquad \ldots(6.2.19)$$

$$(M-b+1 < m \leq M)$$

$$\frac{d}{dt} Q_{m,n}(t) + \mu_1 Q_{m,n}(t) - \lambda_1 Q_{m-1,n}(t) - \lambda_2 Q_{m,n-1}(t) = 0.$$

$$(M-b+1 < m \leq M;\ n \geq 1) \qquad \ldots(6.2.20)$$

Note when a = 1 equation (6.2.13) and (6.2.14) can not occur Let us define the following generating functions,

$$H_m(y,t) = \sum_{n=o}^{\infty} P_{m,n}(t) y^n \quad , \ |y| \leq 1 \ ; \ 0 \leq m \leq a-1$$

$$\ldots(6.2.21)$$

$$F_m(y,t) = \sum_{n=o}^{\infty} Q_{m,n}(t) y^n \quad , \ |y| \leq 1 \ ; \ 1 \leq m \leq M$$

$$\ldots(6.2.22)$$

Writing $\alpha(y) = \lambda_1 + \lambda_2 - \lambda_2 y$

Conbining the equations (6.2.17) to (6.2.18) we get,

$$\sum_{n=o}^{\infty} \frac{d}{dt} Q_{m,n}(t)y^{n} + (\lambda_1+\lambda_2+\mu_1) \sum_{n=o}^{\infty} Q_{m,n}(t)y^{n} - \lambda_1 \sum_{n=o}^{\infty}$$

$$-\lambda_1 \sum_{n=0}^{\infty} Q_{m-1,n}(t)y^{n} - \lambda_2 \sum_{n=1}^{\infty} Q_{m,n-1}(t)y^{n} - \mu_1 \sum_{n=o}^{\infty} Q_{m+b,n}(t) = 0 \; .$$

$$(m\geq 1 \; , \; n\geq 1) \qquad \ldots (6.2.23)$$

The above relation can also be written as

$$\frac{d}{dt} F_m(y,t)+(\alpha+\mu_1)F_m(y,t)-\lambda_1 F_{m-1}(y,t)-\mu_1 F_{m+b}(y,t) = 0 \; .$$

$$(1 \leq m < m-b+1) \qquad \ldots (6.2.24)$$

From equation (6.2.19) and (6.2.20) we have

$$\frac{d}{dt} F_m(y,t)+\mu_1 F_m(y,t)-\lambda_1 F_{m-1}(y,t)-\lambda_2 y F_m(y,t) = 0.$$

$$(M-b+1 \leq m \leq M) \qquad \ldots (6.2.25)$$

From (6.2.15) and (6.2.16) we have

$$\frac{d}{dt} F_o(y,t)-\mu_1 \sum_{k=a}^{b} F_k(y,t)+(\mu_1+\alpha)F_o(y-1)-\lambda_1 H_{a-1}(y,t) = 0.$$

$$\ldots(6.2.26)$$

From (6.2.13) to (6.2.14) we have

$$\frac{d}{dt} H_m(y,t)+ \alpha H_m(y,t)-\lambda_1 H_{m-1}(y,t)-\mu_1 F_m(y,t) = 0,$$

$$(1 \leq m \leq a-1) \qquad \ldots(6.2.27)$$

From (6.2.11) and (6.2.12) we have

$$\frac{d}{dt} H_o(y,t)+[\alpha+\mu_2(1-1/y)]H_o(y,t)- \mu_1 F_o (y,t)$$

$$+ \mu_2 (1/y-1) P_{0,0}(t)=0 . \qquad \ldots(6.2.28.)$$

The difference equation (6.2.24) can be written as

$$h(E)F_{m-1}(y,t) = 0 , m = 1,2,\ldots\ldots\ldots\ldots\ldots M \quad \ldots(6.2.29)$$

Taking Laplace trensfrom of (6.2.29) we get

$$h(E) \bar{F}_{m-1}(y,s) = 0 \qquad \ldots(6.2.30)$$

Thus the characteristic equation becomes

$$h(z)=\mu_1 z^{b+1}-(\alpha+\mu_1+s)z+\lambda_1 = 0. \qquad \ldots(6.2.31)$$

Hence by Rouche's theorem, it can be shown that there will be only one zero of $h(Z)=0$, inside the unit circle $|z|= 1$. Denote this root by $\omega=\omega(y)$ since $\sum_{m=0}^{M} F_m(y,t) < M$,

we must have

$$F_m(y,t) = F_o(y,t)\omega^m, (1 \le m < M-b+1) \qquad \ldots(6.2.32)$$

Now to obtain the value of F_m, $(M-b+1 \le m \le M)$ we take the Laplace transfrom of (6.2.25) then solve it recursively, we get

$$\bar{F}_m(y,s) = \left[\frac{s+\mu_1-\lambda_2 y}{\lambda_1}\right]^{M-m} \bar{F}_M(y,s).$$

$$(M-b+1 \le m \le M) \qquad \ldots(6.2.33)$$

Taking Laplace transfrom of (6.2.27) then solving it, recursively we get

$$\bar{H}_m(y,s)=\left[\frac{\lambda_1}{\alpha+s}\right]^m \bar{H}_o(y,s)+\mu_1\omega^{m+1}\left[\frac{\left(\frac{\mu_1}{\alpha+s}\right)^m-1}{\lambda_1(\alpha+s)\omega}\right]\bar{F}_o(y,s)$$

$$\ldots(6.2.34)$$

Taking Laplace transfrom of the characteristic equation (6.2.31.) and put $z=\omega$, we get

$$\mu_1 = \frac{\lambda_1-(\alpha+s)\omega}{\omega(1-\omega)^b}$$

Putting the value of μ_1 from (6.2.35) in (6.2.34) we get

$$\bar{H}_m(y,s) = [\frac{\lambda_1}{\alpha+s}]^m \bar{H}_o(y,s) + \frac{[\frac{\lambda_1}{\alpha+s}]^m - \omega^m}{(1-\omega)^b} \bar{F}_o(y,s)$$

$$(1 \le m \le a-1) \qquad \text{....(6.2.35)}$$

Now taking Laplace transfrom of (6.2.26) we get.

$$\lambda_1 \bar{H}_{a-1}(y,s) - (\alpha+s+\mu_1)\bar{F}_o(y,s) + \mu_1 \sum_{k=a}^{b} \bar{F}_k(y,s) = 0$$

$$\text{...(6.2.36)}$$

Using (6.2.35) and $\bar{F}_m(y,s) = \bar{F}_o(y,s)\,\omega^m$, $1 \le m < M-b+1$

Substituting for $\bar{F}_k(y,s)$, $k=a,a+1,a+2,\ldots\ldots\ldots b$,and

$\bar{H}_{a-1}(y,s)$ in (6.2.34) we obtain

$$\bar{F}_o(y,s) = \frac{\lambda_1 \omega(1-\omega)(1-\omega^b)\left(\frac{\lambda_1}{\alpha+s}\right)^{a-1}}{\lambda_2(1-y)(\omega^{a+1}-\omega^{b+1}) + \lambda_1(1-\omega)[1-\omega\left(\frac{\lambda_1}{\alpha+s}\right)^{a-1}]}\bar{H}_o(y,s)$$

...(6.2.37)

Taking Laplace transfrom of (6.2.28), we get

$$y\mu_1 \bar{F}_o(y,s)-\{(\alpha+s)y - \mu_2(1-y)\}\bar{H}_o(y,s)=\mu_2(1-y)\bar{P}_{o,o}(s)$$

...(6.2.38)

Putting the value of $\bar{F}_O(y,s)$ from (6.2.37) in (6.2.38) we obtain

$$\bar{H}_o(y,s)=[\lambda_2(1-y)(\omega^{a+1}-\omega^{b+1})+\lambda_1(1-\omega)\{1-\omega\left(\frac{\lambda_1}{\alpha+s}\right)^{a-1}\}].$$

$$\left[\frac{\mu_2(1-y)}{A(y)}\right]\bar{P}_{o,o}(y,s)$$

...(6.2.39)

where

$$A(y,s)=\lambda_1(1-y)[\mu_2-\mu_2\omega-\lambda_2 y][1-\omega\left(\frac{\lambda_1}{\alpha+s}\right)^{a-1}] + \lambda_1\mu_1(1-\omega^b)\omega y$$

$$\{\left(\frac{\lambda_1}{\alpha+s}\right)^{a-1}-1\}-\lambda_2(1-y)[(\alpha+s)y-\mu_2(1-y)][\omega^{a+1}-\omega^{b+1}]$$

...(6.2.40)

Now the joint probability generating funcation of the number of type-1 and type-2 units is given by,

$$\phi(x,y) = \sum_{n=o}^{\infty}\sum_{m=o}^{a-1} P_{m,n}\, y^{n}\, x^{n} + \sum_{n=o}^{\infty}\sum_{m=o}^{M} Q_{m,n}(t)\, y^{n}\, x^{m}$$

$$= H_{o}(y,t) + \sum_{m=1}^{a-1} H_{m}(y,t)\, x^{m} + F_{o}(y,t) + \sum_{m=1}^{M-b} F_{m}(y,t)\, x^{m}$$

$$+ \sum_{m=M-b+1}^{M} F_{m}(y,t)\, x^{m}.$$

Taking Laplace transfrom of the above equation , we get

$$L\phi\,(x,y) = \bar{H}_{o}(y,s) + \sum_{m=1}^{a-1} \bar{H}_{m}(y,s)x^{m} + \bar{F}_{o}(y,s) + \sum_{m=1}^{M-b} \bar{F}_{m}(y,s)x^{m}$$

$$+ \sum_{m=M-b+1}^{M} \bar{F}_{m}(y,s)x^{m}$$

...(6.2.41)

Putting the value of the certain terms in(6.2.41) and solving it,we get

$$L\bar{\phi}\ (x,y)= \frac{\mu_1\ (1-y)P_{o,o}(s)}{A(y)(1-x\omega)\ (\alpha+s-\lambda_1 x)}\ [B(y)-c(y)+D(y)]$$

...(6.2.42)

where

$$B(y)=(\alpha+s)[\lambda_2(1-y)(\omega^{a+1}-\ \omega^{b+1})+\lambda_1(1-\omega)](1-x\omega)[1-(\frac{\lambda_1 x}{\alpha+s})^a\ ]$$

$$+\ \lambda_1\ (1-\omega)(\alpha+s-\lambda_1 x)(\frac{\lambda_1}{\alpha+s})^{a-1}\ (x^a\ \omega^{a+1}\ -\ \omega^{b+1})$$

$$c(y)=\lambda_1\ \omega(1-\omega)(\frac{\lambda_1}{\alpha+s})^{a-1}\ (\omega x)^{M-b+1}\ (1-\ \frac{\lambda_1 x}{\alpha+s}\ -\omega^b\ +\ \omega^b\ \frac{\lambda_1}{\alpha+s})$$

$$D(y)=\frac{(x\omega)^{M-b+1}(1-\omega^b)\{(s+\mu-\lambda_2 y)^b-(x\omega)^b\}(1-x\omega)(\alpha+s-\lambda_1 x)}{\lambda_1^M\ (s+\mu_1-\lambda_2 y-xw)(\alpha+s)}$$

The solution is now complete except for the unknown $P_{o,o}$

Let $\omega(y)\to\ \theta$ as $y\to 1$ .Thus from (6.2.31) we see that

$$\frac{\lambda_1}{\mu_1}=\frac{\theta(1-\theta^b\ )}{1-\theta}$$

Since $\phi\ (1,1)=1$ therefore $L\phi(1,1)=\ 1/s.$

Now

$$\lim_{y\to 1} L\phi\ (x,y) = \lim_{y\to 1} \frac{\mu_2\ (1-y)P_{o,o}}{A(y)(1-x\omega)(\alpha+s-\lambda_1 x)} [B(y)-c(y)+D(y)]. \qquad \ldots(6.2.43)$$

Taking the Limits separately of all the three terms on R.H.S. and denoting by :

$\lim_{y\to 1} L\phi_1\ (1,1), \lim_{y\to 1} L\phi_2(1,1), \lim_{y\to 1} L\phi_3(1,1)$

$$\lim_{y\to 1} L\phi_1\ (1,1) = \frac{(1+s/\lambda_1)(1-\theta)\{1-(\frac{\lambda_1}{\alpha+s})^{a}\}+ (\frac{\lambda_1}{\lambda_1+s})^{a-1} (\theta^{a+1}-\theta^{b+1})}{[(1-\theta)\rho_2)\{1-\theta\ (\frac{\lambda_1}{\alpha+s})^{a-1}\}(a-1)(1-\theta)\rho_2 - \rho_2(1+s/\lambda_1)(\theta^{a+1}-\theta^{b+1})]} \qquad \ldots(i)$$

$$\mathrm{Lim}_{y\to 1} L\theta_2(1,1) = \frac{\theta^{M-b+2}(\frac{\lambda_1}{\lambda_1+s})^{a-1}(1-\frac{\lambda_1}{\lambda_1+s}-\theta^{b}+\theta^{b}\frac{\lambda_1}{\lambda_1+s})}{s\left[(1-\theta-\rho_2)\{1-\theta(\frac{\lambda_1}{\lambda_1+s})^{a-1}\}-\rho_2(a-1)(1-\theta)-\rho_2(1+\frac{s}{\lambda_1})(\theta^{a+1}-\theta^{b+}\right.} \qquad \ldots(ii)$$

and

$$\lim_{y\to 1} L\phi_3(1,1) = -\frac{\theta^{M-b+1}(1-\theta^{b})\{(s+\mu_1-\lambda_2)^{b}-\theta^{b}\}}{\lambda_1^{M+1}(s+\mu_1-\lambda_2-\theta)(\lambda_1+s)[(1-\theta-\rho_2)\{1-\theta(\frac{\lambda_1}{\alpha+s})^{a-1}\} - (a-1)(1-\theta)\rho_2-\rho_2(1+\frac{s}{\lambda_1})(\theta^{a+1}-\theta^{b+1})} \qquad (iii)$$

Taking x=y=1, put the values from (i), (ii), (iii) in (6.2.43) we get

$$\lim_{S\to 0} S\,\bar{P}_{o,o}(s) =$$

$$= \frac{[(1-\theta-\rho_2)\{1-\theta(\frac{\lambda_1}{\alpha+s})^{a-1}-\rho_2(a-1)(1-\theta)-\rho_2(1+s/\lambda_1)(\theta^{a+1}-\theta^{b+1})]}{\left[\begin{array}{l}(1+\frac{\lambda_1}{s})(1-\theta)\quad\{1-(\frac{\lambda_1}{\alpha+s})^{a} + (\frac{\lambda_1}{\alpha+s})^{a-1}(\theta^{a+1}-\theta^{b+1})\\ + \frac{\theta^{M-b+2}}{s}(\frac{\lambda_1}{\lambda_1+s})^{a-1}(1-\frac{\lambda_1}{\lambda_1+s}-\theta^{b}+\theta^{b}\frac{\lambda_1}{\lambda_1+s})\\ + \frac{\theta^{m-b+1}(1-\theta^{b})\{(s+\mu_1-\lambda_2)^{b}-\theta^{b}\}}{\lambda^{M+1}(s+\mu_1-\lambda_2-\theta)\ (\lambda_1+s)}\end{array}\right]}$$

After taking limits of both side of above expression we get

$$\bar{P}_{o,o}(s) = \frac{[(1-\theta)(1-\rho_2-\theta)-(a-1)(1-\theta)\rho_2-\rho_2(\theta^{a+1}-\theta^{b+1})]}{\left[\begin{array}{l}(a(1-\theta)+(\theta^{a+1}-\theta^{b+1})+\theta^{M-b+2}\frac{(1-\theta^{b})}{\lambda_1}+\frac{\theta^{M-b+1}(1-\theta^{b})}{\lambda_1^{M+2}}\\ \quad\frac{\{(\mu_1-\lambda_2)^{b}-\theta^{b}\}}{(\mu_2-\lambda_2-\theta)}]\end{array}\right]}$$

...(6.2.44)

Which is the probability that the idle server finds no unit of either type in the system .Hence the transient state solution exists only if,

$$1-\theta > (1-\theta)\ (\theta+a\rho_2\ ) + \rho_2\ (\theta^{a+1} - \theta^{b+1}) \qquad \ldots(6.2.45)$$

The probability that the server finds no unit of either type in the system or he is busy with a type-2 unit, is given by $\bar{H}_o(1,s)$. To abtain $\bar{H}_o(1,s)$put y=1 in (6.2.39), we get

$$\bar{H}_{o,1}(1.s)=\left[\frac{(1-\theta)^2}{a(1-\theta)+\theta^{a+1}-\theta^{b+1}+\theta^{M-b+2}\dfrac{(1-\theta^b)}{\lambda_1}+\dfrac{\{\theta^{M-b+1}(1-\theta^b)\}}{\lambda_1^{M+2}}} \quad \frac{(\mu_1-\lambda_2)^b-\theta^b}{(\mu_2-\lambda_2-\theta)}\right]$$

$$\ldots(6.2.46)$$

Now the probability that the system contains m units of type-1, no matter how many type-2 units are present when the server is idle, is given by $\bar{H}_m(1,s)$ .To obtain $\bar{H}_m(1,s)$,put y=1 in (6.2.35) we get,

$$\bar{H}_m(1.s) = \bar{H}_o(1,s)\ [\ \frac{1-\theta^{m+1}}{1-\theta}\ ]$$

$$(1 \le m \le a-1) \qquad \ldots(6.2.47)$$

The probability that there one m units of type-1 when the server is busy with a batch of type-1 units is given by $\bar{F}_m(1,s)$ to obtain $\bar{F}_m(1,s)$ we take Laplace transfrom of (6.2.32) then solving for y=1 after putting the value of $\bar{F}\ (y,s)$ we get

$$\bar{F}_m(1,s) = \frac{1-\theta^{b}}{1-\theta}\ \theta^{m+1}\ \ \bar{H}_o(1,s)$$

$$(1 \le m < M-b+1) \qquad \ldots(6.2.48)$$

From (6.2.33) , we get

$$\bar{F}_m(1,s) = (\ \frac{s+\mu_1-\lambda_2}{\lambda_1 zz}\ )^{M-m}\ \bar{F}_M\ (1.s)$$

$$(M-b+1 \le m \le M) \qquad \ldots(6.2.49)$$

6.3 MEAN NUMBER OF THE UNITS IN THE SYSTEM

Difine

L_{q1} :- The mean number of type-1 units in the system

L_{q2} :- The mean number of type-2 units in the sustem

The mean number of type-1 units in the system may be obtained by differentiating the joint probability generating fuction ϕ (x,y) with respect to x and setting x=y=1 or from the partial generating fuction $F_m(y,t)$ and $H_m(y,t)$. By either method we obtain finally,

$$L_{q1} = \sum_{m=o}^{a-1} m \, \bar{H}_m(1,s) + \sum_{m=o}^{M} m \, \bar{F}_m(1,s) \qquad \ldots(6.3.1)$$

Now obtaining the values of two the terms on R.H.S.of (6.3.1) we get

$$\sum_{m=o}^{a-1} m \, \bar{H}_m(1,s) = \bar{H}_o(1,s) \left[\frac{1}{1-\theta} \sum_{m=o}^{a-1} m \, (1-\theta^{m+1}) \right]$$

or

$$\sum_{m=o}^{a-1} m \, \bar{H}_m(1,s) = \bar{H}_o(1.s) \left[\frac{1}{1-\theta} \quad \frac{a(a-1)}{2} \left[1- \frac{\theta^2 \, (1-\theta^{a-1})}{1-\theta}\right]\right] \qquad \ldots(6.3.2)$$

To obtain the value of $\sum\limits_{m=o}^{M} m \bar{F}_m(1,s)$, we write

$$\sum_{m=o}^{M} m \bar{F}_m(1,s) = \sum_{m=1}^{M-b} m \bar{F}_m (1,s) + \sum_{m=M-b+1}^{M} m \bar{F}_m(1,s)$$

...(6.3.3.)

To obtain the value of two terms on R.H.S. of (6.3.3.), we consider

$$\sum_{m=1}^{M-b} m \bar{F}_m (1,s) = \frac{1-\theta^b}{(1-\theta)^3} \theta^2 [-M\theta^M (1-\theta)+(1-\theta^M)]\bar{H}_o (1,s)$$

...(6.3.4)

$$\sum_{m=M-b+1}^{M} m \bar{F}_m(1,s) = \frac{k^{1-m}[\{M-b+1\}k^{M-b} (1-k^b) - k^M b\} (1-k)+k^{M-b+1}(1-k^b)}{(1-k)^2} \bar{F}_m(1,s)$$

...(6.3.5)

where $k = \dfrac{\lambda_1}{s+\mu_1-\lambda_2}$

Putting the values from (6.3.4) and (6.3.5) in (6.3.3)

we get

$$\sum_{m=o}^{M} m\ \bar{F}_m\ (1,s) = \frac{(1-\theta^b)}{(1-\theta)^3}\ \theta^2\ [-M\theta^M\ (1-\theta)+(1-\theta^M)]\bar{H}_o(1,s)$$

$$+\ \frac{k^{1-M}\ [(M-b+1)k^{M-b}(1-k^b)-k^M\ b](1-k)+k^{M-b+1}(1-k^b)\ ]}{(1-k)^2}\ \bar{F}_M(1,s)$$

...(6.3.6)

Putting the values from (6.3.2) and (6.3.6) in (6.3.1) we get

$$L_{q1} = \frac{\bar{H}_o(1,s)}{(1-\theta)}\left[\frac{a(a-1)}{2}\ \{1-\theta^2\ \frac{(1-\theta^{a-1})}{(1-\theta)}\}+\ \frac{\theta\ \lambda_1}{\mu_1(1-\theta)}\ \{-M\ \theta^M(1-\theta)+(1-\theta^M)\}\right]$$

$$+\ \frac{k^{1-m}[\{(M-b+1)k^{M-b}(1-k^b)-k^M\ b\}+k^{M-b+1}(1-k^b)\ ]}{(1-k)^2}\ \bar{F}_M(1,s)$$

...(6.3.7)

If we compare the result (6.3.7) with the mean queue length of $M/M^{a.b}/1$ queueing system dicussed in Medhi [133] or in Bohain and Borthakur [23] it is clear that the mean number (6.3.7) is not changed by the imposition of the preemptive priority discipline

similaly the mean number of type-2 units in the system is given by

$$L_{q2} = \sum_{m=o}^{a-1} \bar{H}_m (1,s) + \sum_{m=o}^{M} \bar{F}_m (1,s) \qquad \ldots(6.3.8)$$

Now of taining the value of the two terms of R.H.S of (6.3.8) we get

$$\sum_{m=o}^{a-1} \bar{H}_m (1,s) = \frac{\bar{H}_o(1,s)}{(1-\theta)^2} [\sum_{m=o}^{a-1} (1-\theta^m) - (1-\theta) \sum_{m=o}^{a-1} m\,\theta^m]$$

$$\ldots(6.3.9)$$

after solving above equation we have

$$\sum_{m=o}^{a-1} \bar{H}_m (1,s) = \frac{\bar{H}_o(1,s)}{(1-\theta)^3} [(1-\theta)\{a-1-\theta(1-\theta^a)\}-(1-\theta^a)]$$

$$\ldots(6.3.10)$$

$$\sum_{m=o}^{M} \bar{F}_m (1.s) = \bar{H}_o (1,s)$$

$$\left[\frac{\{1-(M-b+2)\theta^{M-b+1}-(b+1)\theta^b+(M+2)\theta^M\}(1-\theta)^2 + 2\{\theta-\theta^{M-b+2}-\theta^{b+1}+\theta^{M+2}\}(1-\theta)]}{(1-\theta)^4} \right]$$

$$\ldots(6.3.11)$$

Putting the values from (6.3.9)+(6.3.10) in (6.3.7) we get

$$L_{q_2} = \frac{\bar{H}_0(1,s)}{(1-\theta)^4} \left[\begin{array}{l} (1-\theta)^2 \{a-1-\theta(1-\theta^a)\}-(1-\theta)(1-\theta^a) \\ +\{1-(M-b+2)\theta^{M-b+1}-(b+1)\theta^b+(M+2)\theta^M\}(1-\theta)^2 \\ +2\{\theta-\theta^{M-b+2}-\theta^{b+1}+\theta^{M+2}\}(1-\theta) \end{array} \right].$$

...(6.3.12)

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